A Modular Speed Limit on the Qubit:

Bloch Precession, Saturation, Channel Data-Processing, and the Vanishing of Modular Flow at Pure States

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Abstract

We establish a complete operator-theoretic and operational picture of modular flow on the qubit, building from the GNS construction. For any pair of qubit states ρ_M (modular) and ρ_E (evaluation) with Bloch vectors of magnitudes V_M, r along axes $\hat{n}, \hat{m}$, and rapidity $\eta_M = \operatorname{arctanh}(V_M)$, the variance of the modular Hamiltonian $K_M = -\log \rho_M$ admits the cross-state closed form

$$\mathrm{Var}_E(K_M) \;=\; \eta_M^2\big(1 - r^2(\hat{n}\cdot\hat{m})^2\big),$$

yielding a Mandelstam–Tamm modular speed limit $|d\langle O\rangle/dt| \le 2\eta_M\sqrt{1-r^2\cos^2\theta}\cdot\sqrt{\mathrm{Var}_E(O)}$ whose maximizing-observable saturation ratio is $r\sin\theta/\sqrt{1-r^2\cos^2\theta}$, equal to 1 exactly when $r = 1$. We prove the Heisenberg-picture data-processing inequality $\mathrm{Var}_{\rho_E}(\Phi^*(K_M)) \le \mathrm{Var}_{\Phi(\rho_E)}(K_M)$ for any CPTP map Φ, derive an integrated form yielding the Bures-distance bound $d_B(\rho_E(0), \rho_E(t)) \le 2\eta_M\sqrt{1-r^2\cos^2\theta}\,|t|$, give an operator-norm version $\|[K_M, O]\|_{\mathrm{op}} \le 2\eta_M\|O\|_{\mathrm{op}}$ that saturates for off-axis Pauli operators, and establish a minimum-time relation: for pure ρ_E an orthogonal pure state is reachable in finite modular time iff $\hat{n}\cdot\hat{m} = 0$, in which case $\tau_\perp = \pi/(2\eta_M)$. The Bloch vector of ρ_E precesses around $\hat{n}$ at angular frequency $2\eta_M$, with the work extractable from a half-cycle bounded by $|W| \le 2r\sin\theta\,|\boldsymbol{h}_\perp|$ for any observable Hamiltonian $H = h_0\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}$. The bound vanishes at $V_M \to 1$ when $\rho_E = \rho_M$ despite the divergence of η_M, a finite-dimensional analog of the third law of thermodynamics. The maximum modular speed at fixed ρ_M is $2\eta_M$, attained for pure ρ_E on the equator of the modular axis ($r = 1$, $\theta = \pi/2$). The underlying decomposition $\rho_M^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}\phi(t)}\exp(\mathrm{i}t\eta_M\hat{n}\cdot\boldsymbol{\sigma})$ identifies the modular operator with an $\mathrm{SL}(2,\mathbb{C})$ boost element with imaginary spinor rapidity $\zeta = 2\mathrm{i}t\eta_M$, providing a finite-dimensional analog of the Bisognano–Wichmann theorem in which the universal QFT factor 2π is replaced by the state-dependent $2\eta_M$. Every numerical coefficient is verified to operator-norm precision below 4×10^{-15}.

Keywords: Modular flow; Tomita–Takesaki theory; GNS construction; KMS condition; quantum speed limit; Mandelstam–Tamm bound; rapidity; varentropy; Bisognano–Wichmann theorem; Bloch precession; quantum batteries; data-processing inequality; qubit.
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1 Introduction

1.1 Modular flow and the Bisognano–Wichmann theorem

For a faithful state ω on a von Neumann algebra $\mathcal{M}$, Tomita–Takesaki theory [25,26] associates a one-parameter group of automorphisms $\sigma_t^\omega : \mathcal{M} \to \mathcal{M}$, the *modular flow*. The Bisognano–Wichmann theorem [4,5] identifies the modular flow of the Minkowski vacuum, restricted to a Rindler wedge, with the one-parameter group of Lorentz boosts orthogonal to the wedge edge: at modular time t, the boost rapidity is $2\pi t$. The factor 2π is universal, fixed by the KMS condition at the Unruh temperature.

For finite-dimensional algebras, modular flow is a one-parameter group of inner automorphisms generated by a bounded modular Hamiltonian $K = -\log \rho$ [9, 28]. The natural finite-dimensional analog of Bisognano–Wichmann would identify this flow with a Lorentz-like boost with state-dependent rapidity. We establish such an identification for qubit states.

1.2 Continuous qubit measurement and $\mathrm{SL}(2,\mathbb{C})$

Burns, Greenfield, and Dressel [7] showed that the combined group of unitary evolution and non-unitary measurement backaction on a continuously monitored qubit is precisely $\mathrm{SL}(2,\mathbb{C})$. Under the exceptional isomorphism $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}^{+}(1,3)$, unitary rotations correspond to spatial $\mathrm{SO}(3)$-rotations while measurement backaction corresponds to Lorentz boosts whose rapidity tracks measurement strength. Combined with the Bernoulli Fisher information identity [22]

$$I(V) \;=\; \frac{1}{1-V^2} \;=\; \gamma^2(V), \qquad V = 2p-1 \in (-1,1), \tag{1}$$

the rapidity of a measurement-induced boost equals the inverse hyperbolic tangent of the Bernoulli visibility, $\eta = \operatorname{arctanh}(V)$.

1.3 Contributions

The present paper closes the gap between the algebraic (Bisognano–Wichmann) and operational (Burns–Greenfield–Dressel) sides through twelve numbered theorems (Sections 4–13) plus the third-law analog as Corollary, organized into a single integrated structural picture. A supplementary Tomita-relation theorem (Theorem 1) is presented in Section 3.

1. *Spinor decomposition* (Theorem 2): $\rho^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}\phi(t)} \exp(\mathrm{i}t\eta\hat{n}\cdot\boldsymbol{\sigma})$ with imaginary spinor rapidity $\zeta = 2\mathrm{i}t\eta$.
2. *Cross-state variance* (Theorem 3): $\mathrm{Var}_E(K_M) = \eta_M^2(1 - r^2\cos^2\theta)$.
3. *Modular Mandelstam–Tamm bound with saturation* (Theorems 4–5): saturated by $O = (\hat{n}\times\hat{m})\cdot\boldsymbol{\sigma}$.
4. *Channel data-processing* (Theorem 6): the Heisenberg-picture inequality $\mathrm{Var}_{\rho_E}(\Phi^*(K_M)) \le \mathrm{Var}_{\Phi(\rho_E)}(K_M)$ holds for any CPTP map Φ.
5. *Integrated bound* (Theorem 7): $d_B(\rho_E(0),\rho_E(t)) \le 2\eta_M\sqrt{1-r^2\cos^2\theta}\,|t|$, saturating along precession orbits.
6. *Time-energy uncertainty* (Theorems 8–9): the Bures-distance evolution rate saturates at $2\eta_M\sin\theta$ for pure ρ_E; an orthogonal pure state is reachable in finite modular time iff $\hat{n}\cdot\hat{m} = 0$, in which case $\tau_\perp = \pi/(2\eta_M)$.
7. *Operator-norm bound* (Theorem 10): $\|[K_M, O]\|_{\mathrm{op}} \le 2\eta_M\|O\|_{\mathrm{op}}$, saturating for off-axis Pauli operators.
8. *Bloch precession* (Theorem 11): Bloch vector precesses around $\hat{n}$ at angular frequency $2\eta_M$.
9. *Work extraction* (Theorem 12): $|W| \le 2r\sin\theta\,|\boldsymbol{h}_\perp|$ over a half-cycle.
10. *Maximum modular speed* (Theorem 13): $2\eta_M$, attained for pure ρ_E at $r=1$, $\theta=\pi/2$.
11. *Third-law analog* (Corollary 5): self-speed vanishes as $V \to 1$.

The closed form in Theorem 3 reduces, in the special case $\rho_E = \rho_M$, to the Bernoulli varentropy [13,19]; we attribute this provenance explicitly in Remark 3. The genuinely new contributions are the cross-state formula, the saturation theorem, the channel data-processing inequality, the integrated bound with explicit Bures-distance saturation, the time-energy minimum, the operator-norm version, the Bloch-precession identification, and the work-cost bound.

1.4 Organization

Section 2 fixes notation. Section 3 constructs the modular machinery from the GNS representation. Section 4 proves the spinor decomposition. Section 5 establishes the cross-state variance formula. Section 6 derives the Mandelstam–Tamm bound and its saturation. Section 7 proves channel data-processing. Section 8 derives the integrated bound. Section 9 gives the minimum-time relation. Section 10 establishes the operator-norm version. Section 11 proves Bloch precession and the work bound. Section 12 addresses the third-law analog and rank-deficient states. Section 13 gives the maximum modular speed. Section 14 compares with classical speed limits. Section 15 discusses qubit-specificity. Section 16 reports numerical verification. Section 17 discusses connections to related programs, experimental predictions, limitations, and open questions.

2 Setup

A general qubit state has Bloch vector $\boldsymbol{r} = r\hat{n}$, $r \in [0,1]$, $\hat{n} \in S^2$:

$$\rho \;=\; \tfrac{1}{2}(\mathbb{1} + r\hat{n}\cdot\boldsymbol{\sigma}), \tag{2}$$

with eigenvalues $p = (1+r)/2$ and $1-p = (1-r)/2$. The visibility $V = 2p-1 = r$ coincides with the Bloch radius for a state with definite axis. The rapidity is

$$\eta \;=\; \operatorname{arctanh}(r) \;=\; \tfrac{1}{2}\log\bigl(p/(1-p)\bigr), \tag{3}$$

the Lorentz factor $\gamma(r) = (1-r^2)^{-1/2} = \cosh(\eta)$, and the binary Fisher information $I(r) = 1/(1-r^2) = \gamma^2(r) = \cosh^2(\eta)$ [22].

Definition 1 (Standing assumptions). Throughout this paper, unless explicitly noted otherwise:

1. The *modular state* ρ_M is faithful: $\rho_M > 0$ with $V_M \in (0,1)$. The boundary cases $V_M = 0$ (maximally mixed, $K_M = \log 2 \cdot \mathbb{1}$, so the traceless part $K_M^{(0)} = 0$ and modular flow is trivial) and $V_M = 1$ (pure, K_M unbounded) are addressed in Section 12. Faithfulness is required for the modular Hamiltonian $K_M = -\log\rho_M$ to be bounded and for the Tomita–Takesaki construction of Section 3 to apply directly [25].

2. The *evaluation state* ρ_E is allowed to be any qubit density matrix with Bloch radius $r \in [0,1]$. No faithfulness assumption is placed on ρ_E; pure ρ_E ($r = 1$) is the most informative case for saturation results.

3. The angle $\theta = \angle(\hat{n},\hat{m}) \in [0,\pi]$. Cases $\theta = 0$ and $\theta = \pi$ are handled by continuity; the orthogonal case $\theta = \pi/2$ with $r = 1$ produces the maximum modular speed (Theorem 13).

4. Hermitian observables O are bounded operators on $\mathbb{C}^2$, expressed as $O = \beta\mathbb{1} + \boldsymbol{w}\cdot\boldsymbol{\sigma}$ with $\beta \in \mathbb{R}$ and $\boldsymbol{w} \in \mathbb{R}^3$.

3 The GNS construction and modular machinery

3.1 The Hilbert–Schmidt representation

Let $\mathcal{H} = \mathbb{C}^2$ be the qubit Hilbert space. The Hilbert–Schmidt space $\mathcal{H}_{\mathrm{HS}} = M_2(\mathbb{C})$, equipped with the inner product

$$\langle X, Y\rangle_{\mathrm{HS}} \; = \; \mathrm{Tr}(X^*Y), \tag{4}$$

is a four-dimensional Hilbert space carrying a canonical $*$-representation of $M_2(\mathbb{C})$ acting by left multiplication: $\pi(A)X = AX$ for $A \in M_2(\mathbb{C})$ and $X \in \mathcal{H}_{\mathrm{HS}}$. We additionally introduce the *ρ-weighted inner product* on $M_2(\mathbb{C})$:

$$\langle A, B\rangle_\rho \; := \; \mathrm{Tr}(\rho A^* B), \qquad A, B \in M_2(\mathbb{C}). \tag{5}$$

For faithful ρ, this is positive-definite: $\langle A, A\rangle_\rho = \mathrm{Tr}(\rho A^*A) = \|\rho^{1/2}A\|_{\mathrm{HS}}^2 \geq 0$ with equality iff $A = 0$. The Cauchy–Schwarz inequality on $\langle \cdot, \cdot\rangle_\rho$ yields $|\langle A, B\rangle_\rho|^2 \leq \langle A, A\rangle_\rho \langle B, B\rangle_\rho$, which we use in the Robertson uncertainty proof of Theorem 4.

Definition 2 (GNS vector)*.* For a faithful state ρ with $\rho > 0$, the cyclic vector is $\Omega_\rho = \rho^{1/2} \in \mathcal{H}_{\mathrm{HS}}$.

Lemma 1. *Ω_ρ is unit norm: $\|\Omega_\rho\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\rho) = 1$. The GNS triple $(\mathcal{H}_{\mathrm{HS}}, \pi, \Omega_\rho)$ reproduces $\omega(A) = \mathrm{Tr}(\rho A) = \langle \Omega_\rho, \pi(A)\Omega_\rho\rangle_{\mathrm{HS}}$.*

Proof. $\|\Omega_\rho\|_{\mathrm{HS}}^2 = \mathrm{Tr}(\rho^{1/2}\rho^{1/2}) = \mathrm{Tr}(\rho) = 1$. For the second statement, $\langle \Omega_\rho, \pi(A)\Omega_\rho\rangle = \mathrm{Tr}(\rho^{1/2}A\rho^{1/2}) = \mathrm{Tr}(\rho A) = \omega(A)$ by cyclicity of trace. ☐

3.2 The modular operator

Define the modular operator Δ on $\mathcal{H}_{\mathrm{HS}}$ by left-right multiplication:

$$\Delta(X) \; = \; \rho X \rho^{-1}. \tag{6}$$

Proposition 1. *The modular operator Δ has spectral decomposition on $\mathcal{H}_{\mathrm{HS}}$:*

$$\Delta(E_{ij}) \; = \; \frac{p_i}{p_j} E_{ij} \quad \textit{where} \quad E_{ij} = |i\rangle\langle j|, \; \rho|i\rangle = p_i|i\rangle.$$

For $\rho = \mathrm{diag}(p, 1-p)$ on $\mathbb{C}^2$, the eigenvalues of Δ are $\{1, 1, p/(1-p), (1-p)/p\} = \{1, 1, \mathrm{e}^{2\eta}, \mathrm{e}^{-2\eta}\}$.

Proof. $\Delta(E_{ij}) = \rho E_{ij}\rho^{-1} = (p_i/p_j)E_{ij}$ since $\rho|i\rangle = p_i|i\rangle$ and $\rho^{-1}|j\rangle = p_j^{-1}|j\rangle$. ☐

The modular flow on $\mathcal{H}_{\mathrm{HS}}$ is $\Delta^{\mathrm{i}t}(X) = \rho^{\mathrm{i}t}X\rho^{-\mathrm{i}t}$, which agrees with the algebra automorphism $\sigma_t(X) = \rho^{\mathrm{i}t}X\rho^{-\mathrm{i}t}$ acting through the GNS representation.

3.3 Tomita modular conjugation

Define $J : \mathcal{H}_{\mathrm{HS}} \to \mathcal{H}_{\mathrm{HS}}$ by $J(X) = X^*$ (Hermitian adjoint). J is antiunitary: $J(\alpha X) = \overline{\alpha}J(X)$ and $\langle JX, JY\rangle = \overline{\langle X, Y\rangle}$.

Theorem 1 (Tomita relation)**.** *The Tomita operator $S = J\Delta^{1/2}$ satisfies $S(\pi(A)\Omega_\rho) = \pi(A^*)\Omega_\rho$ for all $A \in M_2(\mathbb{C})$.*

Proof. $\Delta^{1/2}(A\Omega_\rho) = \Delta^{1/2}(A\rho^{1/2}) = \rho^{1/2}A\rho^{-1/2} \cdot \rho^{1/2} = \rho^{1/2}A$, using (6). Then $J(\rho^{1/2}A) = (\rho^{1/2}A)^* = A^*\rho^{1/2} = \pi(A^*)\Omega_\rho$. ☐

3.4 The KMS condition

The state ω is a KMS state at modular inverse-temperature $\beta = 1$ with respect to the modular flow [10,25]. Concretely, for $A, B \in M_2(\mathbb{C})$, the correlation function

$$G^+_{AB}(t) \;=\; \omega(A\sigma_t(B)) \;=\; \mathrm{Tr}(\rho A \rho^{\mathrm{i}t} B \rho^{-\mathrm{i}t}) \tag{7}$$

admits an analytic continuation to the strip $0 \le \operatorname{Im} z \le 1$ satisfying the boundary condition

$$G^+_{AB}(t) \;=\; G^-_{BA}(t+\mathrm{i}) \;=\; \omega(\sigma_{t+\mathrm{i}}(B)A). \tag{8}$$

Proposition 2 (KMS condition)**.** *For any $A, B \in M_2(\mathbb{C})$ and faithful ρ, the correlation function $G^+_{AB}(t)$ extends to a holomorphic function on the strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$, continuous on the closure, with boundary identity (8).*

Proof. Since ρ is finite-dimensional with strictly positive spectrum, $\rho^{\mathrm{i}z}$ is an entire function of z via $\rho^{\mathrm{i}z} = \exp(\mathrm{i}z \log \rho)$. The composition

$$G^+_{AB}(z) \;=\; \mathrm{Tr}\big(\rho A \rho^{\mathrm{i}z} B \rho^{-\mathrm{i}z}\big)$$

is therefore entire in z, hence in particular holomorphic on the open strip and continuous on its closure. We verify the boundary identity (8) directly.

LHS. By cyclicity of trace, using that ρ commutes with $\rho^{-\mathrm{i}t}$,

$$\begin{aligned} G^+_{AB}(t) \;&=\; \mathrm{Tr}\big(\rho A \rho^{\mathrm{i}t} B \rho^{-\mathrm{i}t}\big) \\ &=\; \mathrm{Tr}\big(\rho^{\mathrm{i}t} B \rho^{-\mathrm{i}t} \rho A\big) \;=\; \mathrm{Tr}\big(\rho^{\mathrm{i}t} B \rho \rho^{-\mathrm{i}t} A\big) \;=\; \mathrm{Tr}\big(B \rho \rho^{-\mathrm{i}t} A \rho^{\mathrm{i}t}\big). \end{aligned}$$

RHS. Compute $G^-_{BA}(t+\mathrm{i}) = \mathrm{Tr}\big(\rho\, \rho^{\mathrm{i}(t+\mathrm{i})} B \rho^{-\mathrm{i}(t+\mathrm{i})} A\big)$ using $\rho^{\mathrm{i}(t+\mathrm{i})} = \rho^{\mathrm{i}t-1} = \rho^{\mathrm{i}t}\rho^{-1}$ and $\rho^{-\mathrm{i}(t+\mathrm{i})} = \rho^{1-\mathrm{i}t} = \rho\, \rho^{-\mathrm{i}t}$:

$$\begin{aligned} G^-_{BA}(t+\mathrm{i}) \;&=\; \mathrm{Tr}\big(\rho \cdot \rho^{\mathrm{i}t} \rho^{-1}\, B\, \rho \rho^{-\mathrm{i}t} \cdot A\big) \;=\; \mathrm{Tr}\big(\rho^{\mathrm{i}t} B \rho \rho^{-\mathrm{i}t} A\big) \\ &=\; \mathrm{Tr}\big(B \rho \rho^{-\mathrm{i}t} A \rho^{\mathrm{i}t}\big). \end{aligned}$$

Comparing the two final expressions gives $G^+_{AB}(t) = G^-_{BA}(t+\mathrm{i})$, which is (8). ☐

Remark 1. The KMS condition is verified numerically across multiple operator pairs $(A, B) \in \{(\sigma_x, \sigma_x), (\sigma_x, \sigma_y), (E_{01}, E_{10})\}$ and times $t \in \{0.5, 1.0\}$ to operator-norm precision below 3×10^{-15}. See Section 16, Check 11.

3.5 The modular Hamiltonian

The modular Hamiltonian is $K = -\log \rho$, generating the flow via $\rho^{\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}tK}$. Its expectation value coincides with the von Neumann entropy, a basic identity underlying the first law of entanglement entropy [2]:

$$\omega(K) \;=\; -\mathrm{Tr}(\rho \log \rho) \;=\; S_{\mathrm{vN}}(\rho). \tag{9}$$

The free energy at modular temperature is $F = \omega(K) - S_{\mathrm{vN}}(\rho) = 0$, consistent with ρ being its own thermal state at $\beta = 1$.

Lemma 2 (K generates the modular flow on $\mathcal{H}_{\mathrm{HS}}$)**.** *Let $K^L, K^R : \mathcal{H}_{\mathrm{HS}} \to \mathcal{H}_{\mathrm{HS}}$ denote the left- and right-multiplication operators by K:*

$$K^L(X) \;=\; KX, \qquad K^R(X) \;=\; XK.$$

These commute: $K^L K^R = K^R K^L$ (since K on the left and K on the right act on different sides of X). The modular operator Δ on $\mathcal{H}_{\mathrm{HS}}$ from (6) satisfies

$$\Delta \;=\; \mathrm{e}^{-K^L + K^R} \;=\; \mathrm{e}^{-\mathrm{ad}_K}, \tag{10}$$

where $\mathrm{ad}_K(X) = [K, X]$. *Modular flow is therefore* $\Delta^{\mathrm{i}t}(X) = \mathrm{e}^{-\mathrm{i}t\,\mathrm{ad}_K}(X) = \rho^{\mathrm{i}t} X \rho^{-\mathrm{i}t}$, *with infinitesimal generator*

$$\frac{d}{dt}\sigma_t(X)\bigg|_{t=0} \;=\; -\mathrm{i}[K, X]. \tag{11}$$

Proof. Since $K = -\log\rho$, $\rho = \mathrm{e}^{-K}$ and $\log\rho = -K$. Using that K^L and K^R commute and act independently:

$$\Delta(X) \;=\; \rho X \rho^{-1} \;=\; \mathrm{e}^{-K} X \mathrm{e}^{K} \;=\; \mathrm{e}^{-K^L}\mathrm{e}^{K^R}(X) \;=\; \mathrm{e}^{-K^L+K^R}(X) \;=\; \mathrm{e}^{-\mathrm{ad}_K}(X).$$

At modular time t, $\rho^{\mathrm{i}t} = \exp(\mathrm{i}t\log\rho) = \mathrm{e}^{-\mathrm{i}tK}$, so $\rho^{\mathrm{i}t} X \rho^{-\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}tK^L+\mathrm{i}tK^R}(X) = \mathrm{e}^{-\mathrm{i}t\,\mathrm{ad}_K}(X)$. Differentiating at $t = 0$ gives (11). □

4 Modular flow as imaginary boost

Theorem 2. *For any qubit state* $\rho = \frac{1}{2}(\mathbb{1} + r\hat{n}\cdot\boldsymbol{\sigma})$ *with* $r \in [0, 1)$,

$$\rho^{\mathrm{i}t} \;=\; \mathrm{e}^{\mathrm{i}\phi(t)} \exp\!\big(\mathrm{i}t\eta\,\hat{n}\cdot\boldsymbol{\sigma}\big), \tag{12}$$

where $\phi(t) = (t/2)\log\!\big(p(1-p)\big) = (t/2)\log\!\big((1-r^2)/4\big)$ *and* $\eta = \operatorname{arctanh}(r)$. *(At* $r = 0$, $\eta = 0$, $\phi(t) = -t\log 2$, *and* $\rho^{\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}t\log 2}\mathbb{1}$ *is a pure phase, consistent with* $\rho = \mathbb{1}/2$ *commuting with all observables.) Modulo the global phase,* $\rho^{\mathrm{i}t}$ *is a one-parameter subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *along* $\hat{n}$ *with imaginary spinor rapidity* $\zeta(t) = 2\mathrm{i}t\eta$.

Proof. Step 1: Diagonalization. Choose $U \in \mathrm{SU}(2)$ with $U^*(\hat{n}\cdot\boldsymbol{\sigma})U = \sigma_z$. (Such U exists by the spectral theorem applied to the Hermitian operator $\hat{n}\cdot\boldsymbol{\sigma}$, which has eigenvalues ± 1.) Then

$$U^*\rho U \;=\; \tfrac{1}{2}(\mathbb{1} + r\sigma_z) \;=\; \mathrm{diag}(p, 1-p),$$

where $p = (1 + r)/2$. Since the operation $\rho \mapsto \rho^{\mathrm{i}t}$ commutes with conjugation by unitaries, $U^*\rho^{\mathrm{i}t}U = (U^*\rho U)^{\mathrm{i}t} = \mathrm{diag}(p, 1-p)^{\mathrm{i}t} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}t\log p}, \mathrm{e}^{\mathrm{i}t\log(1-p)})$.

Step 2: Logarithm decomposition. Write

$$\log p \;=\; \tfrac{1}{2}\log\!\big(p(1-p)\big) + \tfrac{1}{2}\log\!\big(p/(1-p)\big), \quad \log(1-p) \;=\; \tfrac{1}{2}\log\!\big(p(1-p)\big) - \tfrac{1}{2}\log\!\big(p/(1-p)\big).$$

Define $\eta = \frac{1}{2}\log\!\big(p/(1-p)\big)$ and $\phi(t) = (t/2)\log\!\big(p(1-p)\big)$. Then

$$U^*\rho^{\mathrm{i}t}U \;=\; \mathrm{e}^{\mathrm{i}\phi(t)}\mathrm{diag}(\mathrm{e}^{\mathrm{i}t\eta}, \mathrm{e}^{-\mathrm{i}t\eta}) \;=\; \mathrm{e}^{\mathrm{i}\phi(t)}\exp(\mathrm{i}t\eta\sigma_z).$$

Step 3: Identification of η *with* $\operatorname{arctanh}(r)$. The standard identity $\operatorname{arctanh}(x) = \frac{1}{2}\log\!\big((1 + x)/(1-x)\big)$ for $x \in (-1, 1)$ gives, with $x = 2p - 1 = r$:

$$\operatorname{arctanh}(r) \;=\; \tfrac{1}{2}\log\!\big((1{+}r)/(1{-}r)\big) \;=\; \tfrac{1}{2}\log\!\big((1{+}(2p{-}1))/(1{-}(2p{-}1))\big) \;=\; \tfrac{1}{2}\log\!\big(p/(1{-}p)\big) \;=\; \eta.$$

Equivalently: $r = \tanh(\eta)$, $p(1-p) = (1-r^2)/4 = \mathrm{sech}^2(\eta)/4$, and $\phi(t) = (t/2)\log\!\big(\mathrm{sech}^2(\eta)/4\big)$.

Step 4: Conjugation back. Since $U(\sigma_z)U^* = \hat{n}\cdot\boldsymbol{\sigma}$ and conjugation commutes with the matrix exponential,

$$\rho^{\mathrm{i}t} \;=\; U(U^*\rho^{\mathrm{i}t}U)U^* \;=\; U\big(\mathrm{e}^{\mathrm{i}\phi(t)}\exp(\mathrm{i}t\eta\sigma_z)\big)U^* \;=\; \mathrm{e}^{\mathrm{i}\phi(t)}\exp(\mathrm{i}t\eta\,\hat{n}\cdot\boldsymbol{\sigma}),$$

establishing (12).

Step 5: Spinor rapidity identification. The standard $\mathrm{SL}(2,\mathbb{C})$ boost-rotation along axis $\hat{n}$ with parameter $\zeta \in \mathbb{C}$ is $B(\zeta, \hat{n}) = \exp((\zeta/2)\hat{n}\cdot\boldsymbol{\sigma})$ [17], with ζ the rapidity (or complex boost angle) that appears identically in the corresponding vector representation $\exp(\zeta\, n\cdot K)$ on Minkowski 4-vectors. Setting $B(\zeta, \hat{n}) = \exp(\mathrm{i}t\eta\,\hat{n}\cdot\boldsymbol{\sigma})$ requires $\zeta/2 = \mathrm{i}t\eta$, i.e., $\zeta = 2\mathrm{i}t\eta$. The factor i corresponds to a Wick rotation: a real ζ corresponds to a real Lorentz boost, while a pure imaginary ζ corresponds to an $\mathrm{SU}(2)$ rotation around $\hat{n}$. The state-dependent factor of 2 relating $t\eta$ in the spinor exponent to the rapidity $\zeta = 2\mathrm{i}t\eta$ is the standard half-angle convention of the spinor double cover $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1, 3)$. □

Remark 2 (Bisognano–Wichmann analog). The Bisognano–Wichmann theorem [4] gives modular rapidity $2\pi t$ for the Minkowski vacuum on a Rindler wedge. Theorem 2 gives modular spinor rapidity $2\mathrm{i}t\eta$ for qubit states. The factor i is the Wick rotation; the universal 2π is replaced by the state-dependent 2η.

Corollary 1 (Off-axis precession)**.** *Choose orthonormal $\hat{e}_1, \hat{e}_2 \in \hat{n}^\perp$ with $\hat{e}_1 \times \hat{e}_2 = \hat{n}$. Define $\sigma_i = \hat{e}_i \cdot \boldsymbol{\sigma}$. Then*

$$\sigma_t(\sigma_1) = \cos(2t\eta)\sigma_1 - \sin(2t\eta)\sigma_2, \quad \sigma_t(\sigma_2) = \sin(2t\eta)\sigma_1 + \cos(2t\eta)\sigma_2, \quad \sigma_t(\hat{n}\cdot\boldsymbol{\sigma}) = \hat{n}\cdot\boldsymbol{\sigma}.$$

Proof. By Theorem 2, $\rho^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}\phi(t)} \exp(\mathrm{i}t\eta\, \hat{n}\cdot\boldsymbol{\sigma})$ and similarly $\rho^{-\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}\phi(t)} \exp(-\mathrm{i}t\eta\, \hat{n}\cdot\boldsymbol{\sigma})$. The phase factors cancel in conjugation:

$$\sigma_t(X) \;=\; \exp(\mathrm{i}t\eta\, \hat{n}\cdot\boldsymbol{\sigma})\, X\, \exp(-\mathrm{i}t\eta\, \hat{n}\cdot\boldsymbol{\sigma}).$$

Explicit BCH calculation. For an orthonormal frame $\{\hat{n}, \hat{e}_1, \hat{e}_2\}$ with $\hat{e}_1 \times \hat{e}_2 = \hat{n}$, define $\sigma_n = \hat{n}\cdot\boldsymbol{\sigma}$, $\sigma_i = \hat{e}_i\cdot\boldsymbol{\sigma}$. The Pauli commutation relation $[\sigma_i, \sigma_j] = 2\mathrm{i}\,\epsilon_{ijk}\sigma_k$ in this frame reads

$$[\sigma_n, \sigma_1] \;=\; 2\mathrm{i}\sigma_2, \quad [\sigma_n, \sigma_2] \;=\; -2\mathrm{i}\sigma_1, \quad [\sigma_n, \sigma_n] \;=\; 0,$$

since $\hat{e}_1 \times \hat{e}_2 = \hat{n}$ implies $\hat{n} \times \hat{e}_1 = \hat{e}_2$ and $\hat{n} \times \hat{e}_2 = -\hat{e}_1$.

Apply the BCH formula $\mathrm{e}^A B \mathrm{e}^{-A} = \sum_{n=0}^{\infty} \mathrm{ad}_A^n(B)/n!$ with $A = \mathrm{i}t\eta\sigma_n$:

$$\begin{aligned}
\mathrm{ad}_{\mathrm{i}t\eta\sigma_n}(\sigma_1) \;&=\; \mathrm{i}t\eta[\sigma_n, \sigma_1] \;=\; \mathrm{i}t\eta(2\mathrm{i}\sigma_2) \;=\; -2t\eta\sigma_2,\\
\mathrm{ad}^2_{\mathrm{i}t\eta\sigma_n}(\sigma_1) \;&=\; \mathrm{i}t\eta[\sigma_n, -2t\eta\sigma_2] \;=\; \mathrm{i}t\eta(-2t\eta)(-2\mathrm{i}\sigma_1) \;=\; -4t^2\eta^2\sigma_1,\\
\mathrm{ad}^3_{\mathrm{i}t\eta\sigma_n}(\sigma_1) \;&=\; \mathrm{i}t\eta[\sigma_n, -4t^2\eta^2\sigma_1] \;=\; \mathrm{i}t\eta(-4t^2\eta^2)(2\mathrm{i}\sigma_2) \;=\; 8t^3\eta^3\sigma_2,\\
\mathrm{ad}^4_{\mathrm{i}t\eta\sigma_n}(\sigma_1) \;&=\; 16t^4\eta^4\sigma_1, \quad \ldots
\end{aligned}$$

Summing:

$$\sigma_t(\sigma_1) \;=\; \sigma_1 \sum_{k=0}^{\infty} \frac{(-1)^k(2t\eta)^{2k}}{(2k)!} - \sigma_2 \sum_{k=0}^{\infty} \frac{(-1)^k(2t\eta)^{2k+1}}{(2k+1)!} \;=\; \cos(2t\eta)\sigma_1 - \sin(2t\eta)\sigma_2.$$

The same calculation with σ_2 in place of σ_1, using $\mathrm{ad}_{\mathrm{i}t\eta\sigma_n}(\sigma_2) = +2t\eta\sigma_1$, yields $\sigma_t(\sigma_2) = \sin(2t\eta)\sigma_1 + \cos(2t\eta)\sigma_2$. The on-axis case is immediate: $\mathrm{ad}_{\mathrm{i}t\eta\sigma_n}(\sigma_n) = 0$ implies $\sigma_t(\sigma_n) = \sigma_n$. ☐

5 The cross-state variance formula

Theorem 3. *Let $\rho_M = \frac{1}{2}(\mathbb{1} + V_M\hat{n}\cdot\boldsymbol{\sigma})$ with axis $\hat{n}$ and visibility $V_M \in (0,1)$, and let $\rho_E = \frac{1}{2}(\mathbb{1} + r\hat{m}\cdot\boldsymbol{\sigma})$ with axis $\hat{m}$ and Bloch radius $r \in [0,1]$. Set $\eta_M = \operatorname{arctanh}(V_M)$ and $\theta = \angle(\hat{n}, \hat{m}) \in [0,\pi]$. Then*

$$\mathrm{Var}_E(K_M) \;=\; \eta_M^2\big(1 - r^2(\hat{n}\cdot\hat{m})^2\big) \;=\; \eta_M^2\big(1 - r^2\cos^2\theta\big). \tag{13}$$

Proof. Step 1: Decomposition of K_M. The modular Hamiltonian decomposes as

$$K_M \;=\; -\log\rho_M \;=\; c_0\mathbb{1} + K_M^{(0)},$$

where $c_0 = -(1/2)\log(p(1-p))$ is the trace-normalized constant (with $p = (1+V_M)/2$) and $K_M^{(0)}$ is traceless. We verify by direct computation: the eigenvalues of K_M are $-\log p$ and $-\log(1-p)$, summing to $-\log(p(1-p)) = 2c_0$, so the trace is $2c_0$, confirming c_0 as the trace-normalized constant. The traceless part has eigenvalues

$$-\log p - c_0 \;=\; -\tfrac{1}{2}\log(p/(1-p)) \;=\; -\eta_M, \qquad -\log(1-p) - c_0 \;=\; +\eta_M.$$

By Theorem 2 and the spectral theorem applied to $\hat{n}\cdot\boldsymbol{\sigma}$ (which has eigenvalues ± 1), we have $K_M^{(0)} = -\eta_M(\hat{n}\cdot\boldsymbol{\sigma})$.

Step 2: Variance is invariant under constant shifts. For any state ρ_E and Hermitian A with $A = c\mathbb{1} + A'$,

$$\mathrm{Var}_E(A) \;=\; \langle A^2\rangle_E - \langle A\rangle_E^2 \;=\; \big(c^2 + 2c\langle A'\rangle + \langle A'^2\rangle\big) - \big(c + \langle A'\rangle\big)^2 \;=\; \langle A'^2\rangle - \langle A'\rangle^2 \;=\; \mathrm{Var}_E(A').$$

Therefore $\mathrm{Var}_E(K_M) = \mathrm{Var}_E(K_M^{(0)})$.

Step 3: Mean of $K_M^{(0)}$. Using $\rho_E = \frac{1}{2}(\mathbb{1} + r\hat{m}\cdot\boldsymbol{\sigma})$ and $\mathrm{Tr}(\sigma_i) = 0$, $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$:

$$\langle K_M^{(0)}\rangle_E \;=\; -\eta_M \mathrm{Tr}\big(\rho_E(\hat{n}\cdot\boldsymbol{\sigma})\big) \;=\; -\eta_M\cdot\tfrac{1}{2}\mathrm{Tr}\big(\mathbb{1}\cdot\hat{n}\cdot\boldsymbol{\sigma} + r(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})\big).$$

The first trace vanishes ($\mathrm{Tr}\sigma_i = 0$). For the second, $(\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma}) = (\hat{m}\cdot\hat{n})\mathbb{1} + \mathrm{i}(\hat{m}\times\hat{n})\cdot\boldsymbol{\sigma}$ by the Pauli product identity, so $\mathrm{Tr}\big((\hat{m}\cdot\boldsymbol{\sigma})(\hat{n}\cdot\boldsymbol{\sigma})\big) = 2(\hat{m}\cdot\hat{n})$. Therefore

$$\langle K_M^{(0)}\rangle_E \;=\; -\eta_M\cdot\tfrac{1}{2}\cdot 2r(\hat{m}\cdot\hat{n}) \;=\; -\eta_M r(\hat{n}\cdot\hat{m}).$$

Step 4: Mean square of $K_M^{(0)}$. Since $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$ (Pauli identity for unit vector $\hat{n}$):

$$\langle (K_M^{(0)})^2\rangle_E \;=\; \eta_M^2\mathrm{Tr}\big(\rho_E(\hat{n}\cdot\boldsymbol{\sigma})^2\big) \;=\; \eta_M^2\mathrm{Tr}(\rho_E) \;=\; \eta_M^2.$$

Step 5: Combine.

$$\mathrm{Var}_E(K_M) \;=\; \eta_M^2 - \eta_M^2 r^2(\hat{n}\cdot\hat{m})^2 \;=\; \eta_M^2\big(1 - r^2\cos^2\theta\big). \qquad \square$$

Corollary 2 (Self-variance). *Setting $\rho_E = \rho_M$ ($\theta = 0$, $r = V_M$):*

$$\mathrm{Var}_M(K_M) \;=\; \eta_M^2(1 - V_M^2) \;=\; \eta_M^2/\gamma^2(V_M) \;=\; \eta_M^2\mathrm{sech}^2(\eta_M). \tag{14}$$

Remark 3 (Varentropy provenance). The expression $\mathrm{Var}_M(K_M) = p(1-p)(\log(p/(1-p)))^2$ is the varentropy of a Bernoulli random variable [13, 19], where the varentropy is $V(P) = \mathrm{Var}[-\log P(X)]$. The rapidity-coordinate form η_M^2/γ^2 is an algebraic re-expression of this known information-theoretic quantity. The new content of Theorem 3 is the cross-state extension to $\rho_E \neq \rho_M$, with the angular factor $\cos^2\theta$, which has no direct varentropy precedent.

Corollary 3 (Maximum self-variance). *$\mathrm{Var}_M(K_M)$ as a function of $\eta_M \in [0,\infty)$ attains its maximum at $\eta_{M*}\tanh(\eta_{M*}) = 1$, numerically $\eta_{M*} \approx 1.19968$, $V_{M*} \approx 0.83356$, $\mathrm{Var}_* \approx 0.43923$.*

Proof. $d/d\eta_M[\eta_M^2\mathrm{sech}^2\eta_M] = 2\eta_M\mathrm{sech}^2\eta_M(1 - \eta_M\tanh\eta_M)$, vanishing at the unique positive root. $\square$

Remark 4 (Edge cases of axis alignment). The formula (13) extends continuously to the degenerate cases:

1. *Parallel axes* ($\theta = 0$, $\hat{n} = \hat{m}$): $\cos^2\theta = 1$, giving $\mathrm{Var}_E(K_M) = \eta_M^2(1 - r^2)$. In this case ρ_M and ρ_E commute (both diagonal in the $\hat{n}$-basis), so $\rho_M^{\mathrm{i}t}$ commutes with ρ_E, and $\rho_E(t) = \rho_M^{\mathrm{i}t}\rho_E\rho_M^{-\mathrm{i}t} = \rho_E$ is constant. Therefore $\langle O\rangle_E(t) = \mathrm{Tr}(\rho_E O)$ is constant, $d\langle O\rangle_E/dt = 0$ for every observable O, and the Mandelstam–Tamm bound holds trivially with LHS zero.

2. *Antiparallel axes* ($\theta = \pi$, $\hat{n} = -\hat{m}$): $\cos^2\theta = 1$, same closed form. The states still commute; same conclusion.

3. *Maximally mixed evaluation state* ($r = 0$, $\rho_E = \mathbb{1}/2$): $\mathrm{Var}_E(K_M) = \eta_M^2$, independent of θ. This is the maximum cross-state variance over all ρ_E at fixed ρ_M, attained whenever $r = 0$ or $\theta = \pi/2$.

The closed form (13) therefore extends to all $r \in [0,1]$ and $\theta \in [0,\pi]$ as a single analytic expression, with the apparent singularities at $\sin\theta = 0$ in subsequent results (Theorem 5, $\hat{k} = \hat{n}\times\hat{m}/\sin\theta$) being removable: in the parallel/antiparallel case, the maximizing direction $\hat{k}$ is undefined because the entire orthogonal sector to $\hat{n}$ achieves the same rate, but the rate itself is zero so the limit is well-posed.

6 The modular Mandelstam–Tamm bound and saturation

Theorem 4 (Modular Mandelstam–Tamm)**.** *Under the hypotheses of Theorem 3, for any observable O on $\mathbb{C}^2$ and any t,*

$$\bigl|d\langle O\rangle_E/dt\bigr| \;\le\; 2\eta_M\sqrt{1-r^2\cos^2\theta}\cdot\sqrt{\mathrm{Var}_{\rho_E(t)}(O)}, \tag{15}$$

where $\langle O\rangle_E(t) = \mathrm{Tr}(\rho_E(t)O) = \mathrm{Tr}(\rho_E\sigma_{-t}(O))$, $\rho_E(t) = \rho_M^{\mathrm{i}t}\rho_E\rho_M^{-\mathrm{i}t}$, $\sigma_t(X) = \rho_M^{\mathrm{i}t}X\rho_M^{-\mathrm{i}t}$, and r, θ are the Bloch parameters of ρ_E (equivalently of $\rho_E(t)$, since the component $\cos\theta = \hat{n}\cdot\hat{m}$ is conserved under modular flow). The two forms are related by cyclicity of the trace: $\mathrm{Tr}(\rho_M^{\mathrm{i}t}\rho_E\rho_M^{-\mathrm{i}t}\cdot O) = \mathrm{Tr}(\rho_E\cdot\rho_M^{-\mathrm{i}t}O\rho_M^{\mathrm{i}t}) = \mathrm{Tr}(\rho_E\sigma_{-t}(O))$. At $t=0$, the right-hand side reduces to $2\eta_M\sqrt{1-r^2\cos^2\theta}\cdot\sqrt{\mathrm{Var}_{\rho_E}(O)}$.

Proof. Step 1: Modular Heisenberg derivative. For $\sigma_t(O) = \rho_M^{\mathrm{i}t}O\rho_M^{-\mathrm{i}t}$, using $\log\rho_M = -K_M$ and $d/dt\,\rho_M^{\mathrm{i}t} = \mathrm{i}(\log\rho_M)\rho_M^{\mathrm{i}t} = -\mathrm{i}K_M\rho_M^{\mathrm{i}t}$:

$$\frac{d\sigma_t(O)}{dt} \;=\; -\mathrm{i}K_M\rho_M^{\mathrm{i}t}O\rho_M^{-\mathrm{i}t} + \mathrm{i}\rho_M^{\mathrm{i}t}O\rho_M^{-\mathrm{i}t}K_M \;=\; -\mathrm{i}[K_M,\sigma_t(O)].$$

Since $\langle O\rangle_E(t) = \mathrm{Tr}(\rho_E\sigma_{-t}(O))$ and $d\sigma_{-t}(O)/dt = +\mathrm{i}[K_M,\sigma_{-t}(O)]$ by the chain rule, we obtain

$$\frac{d\langle O\rangle_E}{dt} \;=\; \mathrm{i}\mathrm{Tr}\bigl(\rho_E[K_M,\sigma_{-t}(O)]\bigr).$$

The Mandelstam–Tamm bound depends only on the modulus $|d\langle O\rangle_E/dt|$, so the sign convention does not affect the final inequality.

Step 2: Schrödinger–Robertson uncertainty. For Hermitian A, B and any state ρ, we prove the Schrödinger uncertainty relation [21], which strengthens Robertson's [20]:

$$\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \;\ge\; \tfrac{1}{4}\bigl|\mathrm{Tr}(\rho[A,B])\bigr|^2 + \tfrac{1}{4}\bigl|\mathrm{Tr}(\rho\{\widetilde{A},\widetilde{B}\})\bigr|^2, \tag{16}$$

where $\widetilde{A} = A - \langle A\rangle_\rho\mathbb{1}$ and $\{\cdot,\cdot\}$ is the symmetrized anticommutator. Robertson's inequality (17) below follows by dropping the anticommutator term.

Proof of (16). Define $\widetilde{A},\widetilde{B}$ Hermitian and centred on ρ: $\langle\widetilde{A}\rangle_\rho = \langle\widetilde{B}\rangle_\rho = 0$. Then $[A,B] = [\widetilde{A},\widetilde{B}]$ and $\mathrm{Var}_\rho(A) = \langle\widetilde{A}^2\rangle_\rho$. Apply Cauchy–Schwarz on the ρ-weighted inner product (5):

$$\bigl|\langle\widetilde{A},\widetilde{B}\rangle_\rho\bigr|^2 \;=\; \bigl|\mathrm{Tr}(\rho\widetilde{A}\widetilde{B})\bigr|^2 \;\le\; \langle\widetilde{A},\widetilde{A}\rangle_\rho\langle\widetilde{B},\widetilde{B}\rangle_\rho \;=\; \mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B).$$

(Recall $\langle X,Y\rangle_\rho = \mathrm{Tr}(\rho X^*Y)$, and $X^* = X$ for Hermitian.) Decompose $\widetilde{A}\widetilde{B} = \frac{1}{2}\{\widetilde{A},\widetilde{B}\} + \frac{1}{2}[\widetilde{A},\widetilde{B}]$, where $\{\widetilde{A},\widetilde{B}\}$ is Hermitian and $[\widetilde{A},\widetilde{B}]$ is anti-Hermitian. Then $\mathrm{Tr}(\rho\{\widetilde{A},\widetilde{B}\})$ is real and $\mathrm{Tr}(\rho[\widetilde{A},\widetilde{B}])$ is purely imaginary. Therefore

$$\bigl|\mathrm{Tr}(\rho\widetilde{A}\widetilde{B})\bigr|^2 \;=\; \tfrac{1}{4}\bigl(\mathrm{Tr}(\rho\{\widetilde{A},\widetilde{B}\})\bigr)^2 + \tfrac{1}{4}\bigl|\mathrm{Tr}(\rho[A,B])\bigr|^2.$$

Combining yields (16). The corollary

$$\bigl|\mathrm{Tr}(\rho[A,B])\bigr|^2 \;\le\; 4\mathrm{Var}_\rho(A)\mathrm{Var}_\rho(B) \tag{17}$$

(Robertson's inequality) follows by dropping the non-negative anticommutator term in (16).

Step 3: Apply Robertson at $t=0$. Setting $A = K_M$ and $B = O$ in (17) at state ρ_E:

$$\bigl|d\langle O\rangle_E/dt|_{t=0}\bigr|^2 \;=\; \bigl|\mathrm{Tr}(\rho_E[K_M,O])\bigr|^2 \;\le\; 4\mathrm{Var}_{\rho_E}(K_M)\mathrm{Var}_{\rho_E}(O) \;=\; 4\eta_M^2(1-r^2\cos^2\theta)\mathrm{Var}_{\rho_E}(O).$$

Taking square roots gives (15) at $t=0$.

Step 4: General t by reparameterization. For general t_0, observe $\rho_E(t_0) = \rho_M^{\mathrm{i}t_0}\rho_E\rho_M^{-\mathrm{i}t_0}$ is a valid state. Modular flow has the semigroup property $\rho_M^{\mathrm{i}(t_0+t')} = \rho_M^{\mathrm{i}t_0}\rho_M^{\mathrm{i}t'}$, so $\rho_E(t_0+t') = \rho_M^{\mathrm{i}t'}\rho_E(t_0)\rho_M^{-\mathrm{i}t'}$. Therefore $d\langle O\rangle/dt|_{t=t_0} = d\langle O\rangle_{\rho_E(t_0)}/dt'|_{t'=0}$. Applying Step 3 to the state $\rho_E(t_0)$ in place of ρ_E:

$$\left|d\langle O\rangle_E/dt|_{t=t_0}\right|^2 \;\le\; 4\mathrm{Var}_{\rho_E(t_0)}(K_M)\mathrm{Var}_{\rho_E(t_0)}(O).$$

The Bloch radius of $\rho_E(t_0)$ equals r (unitary invariance of eigenvalues). The Bloch vector $\boldsymbol{m}(t_0)$ satisfies $\hat{n}\cdot\boldsymbol{m}(t_0) = \hat{n}\cdot\boldsymbol{m}$: by direct calculation in the Schrödinger picture, $d/dt[\mathrm{Tr}(\rho_E(t)(\hat{n}\cdot\boldsymbol{\sigma}))] = \mathrm{Tr}\big(-\mathrm{i}[K_M,\rho_E(t)](\hat{n}\cdot\boldsymbol{\sigma})\big) = \mathrm{i}\mathrm{Tr}(\rho_E(t)[K_M,\hat{n}\cdot\boldsymbol{\sigma}]) = \mathrm{i}\mathrm{Tr}(\rho_E(t)(-\eta_M)[\hat{n}\cdot\boldsymbol{\sigma},\hat{n}\cdot\boldsymbol{\sigma}]) = 0$, so $\hat{n}\cdot\boldsymbol{m}(t)$ is conserved. Therefore $\mathrm{Var}_{\rho_E(t_0)}(K_M) = \eta_M^2(1-r^2\cos^2\theta)$ for all t_0 (Theorem 3 applied to $\rho_E(t_0)$), establishing (15). ☐

Theorem 5 (Maximizing-observable saturation ratio)**.** *Suppose $r > 0$ and $\theta \in (0,\pi)$. The bound (15) achieves the value*

$$\left|d\langle O_\star\rangle_E/dt\right| \;=\; 2\eta_M\, r\sin\theta\cdot|\alpha|, \qquad O_\star = \alpha\big(\hat{n}\times\hat{m}/|\hat{n}\times\hat{m}|\big)\cdot\boldsymbol{\sigma}, \quad \alpha\in\mathbb{R}\setminus\{0\}, \tag{18}$$

which is, for $r < 1$, the unique direction of O (up to sign and scale) maximizing the rate of change among all observables with the same $\mathrm{Var}_E(O)$. At $r = 1$ the maximizer is degenerate: every $O = \alpha(\boldsymbol{w}\cdot\boldsymbol{\sigma})$ with $\boldsymbol{w}\cdot\hat{k}\neq 0$ achieves the same saturation ratio. The corresponding ratio is

$$\frac{|d\langle O_\star\rangle_E/dt|}{2\eta_M\sqrt{1-r^2\cos^2\theta}\cdot\sqrt{\mathrm{Var}_E(O_\star)}} \;=\; \frac{r\sin\theta}{\sqrt{1-r^2\cos^2\theta}} \;\le\; 1, \tag{19}$$

with equality (exact saturation) if and only if $r = 1$.

Proof. Let $\hat{k} = \hat{n}\times\hat{m}/|\hat{n}\times\hat{m}|$ (unit vector orthogonal to both $\hat{n}$ and $\hat{m}$). Set $O = \alpha(\hat{k}\cdot\boldsymbol{\sigma})$.

Step 1: Variance of O. Since $\hat{k}\perp\hat{m}$, $\langle\hat{k}\cdot\boldsymbol{\sigma}\rangle_E = r(\hat{k}\cdot\hat{m}) = 0$, and $(\hat{k}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$. Thus $\mathrm{Var}_E(O) = \alpha^2\langle(\hat{k}\cdot\boldsymbol{\sigma})^2\rangle_E - 0 = \alpha^2$.

Step 2: Commutator. Using the Pauli identity $[\hat{a}\cdot\boldsymbol{\sigma},\hat{b}\cdot\boldsymbol{\sigma}] = 2\mathrm{i}(\hat{a}\times\hat{b})\cdot\boldsymbol{\sigma}$ for unit vectors:

$$[K_M^{(0)}, O] \;=\; -\eta_M\alpha[\hat{n}\cdot\boldsymbol{\sigma},\hat{k}\cdot\boldsymbol{\sigma}] \;=\; -2\mathrm{i}\eta_M\alpha(\hat{n}\times\hat{k})\cdot\boldsymbol{\sigma}.$$

Step 3: Explicit evaluation of $\hat{n}\times\hat{k}$. Apply the BAC–CAB identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ with $\boldsymbol{a} = \boldsymbol{b} = \hat{n}$, $\boldsymbol{c} = \hat{m}$:

$$\hat{n}\times(\hat{n}\times\hat{m}) \;=\; \hat{n}(\hat{n}\cdot\hat{m}) - \hat{m}(\hat{n}\cdot\hat{n}) \;=\; \cos\theta\cdot\hat{n} - \hat{m},$$

since $\hat{n}\cdot\hat{n} = 1$ and $\hat{n}\cdot\hat{m} = \cos\theta$. Dividing by $|\hat{n}\times\hat{m}| = \sin\theta$:

$$\hat{n}\times\hat{k} \;=\; (\cos\theta\cdot\hat{n} - \hat{m})/\sin\theta.$$

Step 4: Evaluation expectation. Since $\langle\hat{n}\cdot\boldsymbol{\sigma}\rangle_E = r\cos\theta$ and $\langle\hat{m}\cdot\boldsymbol{\sigma}\rangle_E = r$ (because ρ_E has Bloch vector $r\hat{m}$):

$$\langle(\hat{n}\times\hat{k})\cdot\boldsymbol{\sigma}\rangle_E \;=\; \frac{\cos\theta\cdot r\cos\theta - r}{\sin\theta} \;=\; \frac{-r(1-\cos^2\theta)}{\sin\theta} \;=\; -r\sin\theta.$$

Step 5: Rate of change. Combining Steps 2 and 4:

$$\left|\langle[K_M, O]\rangle_E\right| \;=\; 2\eta_M\alpha\cdot r\sin\theta, \qquad \left|d\langle O\rangle_E/dt\right| \;=\; 2\eta_M\alpha\, r\sin\theta.$$

Step 6: Ratio computation. The bound for this O is $2\eta_M\sqrt{1-r^2\cos^2\theta}\cdot\alpha$. Therefore the ratio in (19) equals $r\sin\theta/\sqrt{1-r^2\cos^2\theta}$. The condition for equality is $r^2\sin^2\theta = 1 - r^2\cos^2\theta$, equivalent to $r^2(\sin^2\theta+\cos^2\theta) = 1$, i.e., $r = 1$.

Step 7: Maximization argument. We show that $O = \alpha(\hat{k}\cdot\boldsymbol{\sigma})$ achieves the maximum of $|d\langle O\rangle_E/dt|^2/\mathrm{Var}_E(O)$ over all traceless observables. Parameterize a generic traceless observable $O = \alpha(\boldsymbol{w}\cdot\boldsymbol{\sigma})$ with $|\boldsymbol{w}| = 1$ and $\alpha > 0$. By the Pauli commutation identity and the same Bloch-trace calculation as in Steps 2–5 (with $\boldsymbol{w}$ in place of $\hat{k}$),

$$\begin{aligned} \mathrm{Var}_E(O) \;&=\; \alpha^2\big(1 - r^2(\boldsymbol{w}\cdot\hat{m})^2\big), \\ \big|d\langle O\rangle_E/dt\big|^2 \;&=\; 4\eta_M^2\alpha^2 r^2\big(\boldsymbol{w}\cdot(\hat{n}\times\hat{m})\big)^2 \;=\; 4\eta_M^2\alpha^2 r^2\sin^2\theta\,(\boldsymbol{w}\cdot\hat{k})^2. \end{aligned}$$

The ratio is

$$R(\boldsymbol{w}) \;:=\; \frac{|d\langle O\rangle_E/dt|^2}{4\eta_M^2\,\mathrm{Var}_E(O)\big(1 - r^2\cos^2\theta\big)} \;=\; \frac{r^2\sin^2\theta\,(\boldsymbol{w}\cdot\hat{k})^2}{\big(1 - r^2(\boldsymbol{w}\cdot\hat{m})^2\big)\big(1 - r^2\cos^2\theta\big)},$$

to be maximized over $|\boldsymbol{w}| = 1$. Decompose $\boldsymbol{w} = a\hat{n} + b\hat{e}_1 + c\hat{k}$ with $a^2 + b^2 + c^2 = 1$, where $\hat{e}_1 = (\hat{m} - \cos\theta\,\hat{n})/\sin\theta$ and $\hat{k} = \hat{n}\times\hat{m}/\sin\theta$. Then $\boldsymbol{w}\cdot\hat{k} = c$ and $\boldsymbol{w}\cdot\hat{m} = a\cos\theta + b\sin\theta$. To maximize R, maximize c^2, i.e., $c = \pm 1$, $a = b = 0$, giving $\boldsymbol{w} = \pm\hat{k}$ and $\boldsymbol{w}\cdot\hat{m} = 0$. Then $\mathrm{Var}_E(O) = \alpha^2$, and

$$R_{\max} \;=\; \frac{r^2\sin^2\theta}{1 - r^2\cos^2\theta}.$$

The saturation ratio in (19) is $\sqrt{R_{\max}} = r\sin\theta/\sqrt{1 - r^2\cos^2\theta}$, matching the direct computation in Steps 1–6. □

Remark 5 (Numerical verification of the saturation ratio). The ratio $r\sin\theta/\sqrt{1 - r^2\cos^2\theta}$ has been verified across $r \in \{0.3, 0.5, 0.7, 0.9, 1.0\}$ and $\theta \in \{30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ\}$ to operator-norm precision below 10^{-15}. At $r = 1, \theta = 90^\circ$: ratio $= 1.0000$ exactly. At $r = 0.7, \theta = 90^\circ$: ratio $= 0.7000$. See Section 16, Check 6.

7 Channel data-processing for the modular variance

Theorem 6 (Channel data-processing for the modular variance)**.** *Let $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a CPTP (completely positive trace-preserving) map with dual Φ^* in the Heisenberg picture. Then*

$$\mathrm{Var}_{\rho_E}\big(\Phi^*(K_M)\big) \;\le\; \mathrm{Var}_{\Phi(\rho_E)}(K_M). \tag{20}$$

That is, the variance of the noised modular Hamiltonian $\Phi^*(K_M)$ in ρ_E is bounded above by the variance of K_M in the noised state $\Phi(\rho_E)$.

Operational reading. *If the readout observable for the speed limit is noised by Φ^* before measurement, the resulting bound is no larger than the original Mandelstam–Tamm bound evaluated in the noised state $\Phi(\rho_E)$. This is the Heisenberg-picture data-processing inequality for the modular Mandelstam–Tamm bound.*

Caveat. *Unlike the operator norm $\|[K_M, O]\|_{\mathrm{op}}$ which contracts under any unital completely positive map, the cross-state variance $\mathrm{Var}_{\rho_E}(K_M)$ is concave in ρ_E and is therefore* not *a Petz monotone function of the state; the bound (20) expresses the correct data-processing statement at the level of the noised modular Hamiltonian.*

Proof. Step 1: Kadison–Schwarz inequality. For any 2-positive unital map Ψ on $M_n(\mathbb{C})$ and Hermitian A,

$$\Psi(A^2) \;\ge\; \Psi(A)^2 \tag{21}$$

in the operator order (Kadison 1952 [12]). Every CPTP map Φ has dual Φ^* that is unital ($\Phi^*(\mathbb{1}) = \mathbb{1}$) and completely positive (hence 2-positive) by the Stinespring dilation theorem [24]; (21) applies with $\Psi = \Phi^*$.

Step 2: Variance contraction. For any state ρ and Hermitian A, compute

$$\begin{aligned}\mathrm{Var}_\rho(\Phi^*(A)) &= \mathrm{Tr}\big(\rho\,\Phi^*(A)^2\big) - \mathrm{Tr}\big(\rho\,\Phi^*(A)\big)^2\\ &\le \mathrm{Tr}\big(\rho\,\Phi^*(A^2)\big) - \mathrm{Tr}\big(\rho\,\Phi^*(A)\big)^2 \quad \text{by (21)}\\ &= \mathrm{Tr}\big(\Phi(\rho)\,A^2\big) - \mathrm{Tr}\big(\Phi(\rho)\,A\big)^2 \quad \text{by Heisenberg–Schrödinger duality}\\ &= \mathrm{Var}_{\Phi(\rho)}(A).\end{aligned}$$

Step 3: Apply to modular Hamiltonian. Set $A = K_M$ and $\rho = \rho_E$. Then $\mathrm{Var}_{\rho_E}(\Phi^*(K_M)) \le \mathrm{Var}_{\Phi(\rho_E)}(K_M)$, which is exactly (20).

Remark on alternative formulations. The reverse statement $\mathrm{Var}_{\Phi(\rho_E)}(\Phi^*(K_M)) \le \mathrm{Var}_{\rho_E}(K_M)$ does *not* hold for general CPTP Φ; counterexamples exist with strict violation (e.g., a unitarily evolved channel preceded by partial depolarization on a state with high coherence). What *does* hold universally is (20): the variance of the noised observable in the original state is bounded by the variance of the unaltered observable in the noised state. This is the precise Heisenberg-picture data-processing inequality, in line with the general theory of monotone metrics on the state space [14, 18]. ☐

Corollary 4 (Data-processing under depolarization). *Under the depolarizing channel* $\Phi_\lambda(\rho) = (1-\lambda)\rho + \lambda\nVdash/2$ *for* $\lambda \in [0,1]$, *with dual* $\Phi^*_\lambda(A) = (1-\lambda)A + \lambda\mathrm{Tr}(A)/2 \cdot \nVdash$,

$$\mathrm{Var}_{\rho_E}\big(\Phi^*_\lambda(K_M)\big) \;\le\; \mathrm{Var}_{\Phi_\lambda(\rho_E)}(K_M),$$

with strict inequality for $\lambda \in (0,1)$ *except in degenerate cases.*

Proof. Φ_λ is a CPTP map for $\lambda \in [0,1]$. Apply Theorem 6. ☐

Remark 6 (Variance is concave in the state). The cross-state variance $\rho_E \mapsto \mathrm{Var}_{\rho_E}(K_M)$ is concave in ρ_E (since $\mathrm{Tr}(\rho_E K_M^2)$ is linear while $\mathrm{Tr}(\rho_E K_M)^2$ is convex). Consequently $\mathrm{Var}_{\Phi_\lambda(\rho_E)}(K_M) \ge (1-\lambda)\mathrm{Var}_{\rho_E}(K_M) + \lambda\mathrm{Var}_{\nVdash/2}(K_M)$ for the depolarizing channel, and in particular the inequality $\mathrm{Var}_{\Phi(\rho_E)}(K_M) \le \mathrm{Var}_{\rho_E}(K_M)$ does *not* hold for general CPTP Φ. Theorem 6 expresses the correct Heisenberg-picture data-processing statement at the level of the noised modular Hamiltonian $\Phi^*(K_M)$.

8 Integrated bound and Bures-distance arclength

Theorem 7 (Integrated bound). *Define the geodesic Bures distance (Bures angle) [6, 27] between two states* ρ_1, ρ_2 *by* $d_B(\rho_1,\rho_2) := 2\arccos\big(\mathrm{Tr}\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\big)$, *the length of the shortest path between the states in the Bures metric (Fisher–Rao information geometry on the state space). Under modular flow generated by* ρ_M,

$$d_B\big(\rho_E(0), \rho_E(t)\big) \;\le\; 2\eta_M\sqrt{1 - r^2\cos^2\theta}\cdot|t|, \qquad \rho_E(t) = \rho_M^{\mathrm{i}t}\rho_E\rho_M^{-\mathrm{i}t}. \tag{22}$$

For pure ρ_E *on the equator of the modular axis* ($r = 1, \theta = \pi/2$), *the bound saturates exactly:* $d_B(\rho_E(0), \rho_E(t)) = 2\eta_M|t|$ *for* $|t| \le \pi/(2\eta_M)$.

Proof. Step 1: Pure-state derivation of Anandan–Aharonov rate. For a pure state $|\psi(t)\rangle = U_t|\psi(0)\rangle$ with $U_t = \exp(-\mathrm{i}tH)$, the Fubini–Study angle $\alpha_{\mathrm{FS}}(t) = \arccos|\langle\psi(0)|\psi(t)\rangle|$ on projective Hilbert space satisfies $|d\alpha_{\mathrm{FS}}/dt| \le \sqrt{\mathrm{Var}_E(H)}$ (Anandan–Aharonov [1]). The geodesic Bures distance for pure states equals twice the Fubini–Study angle, $d_B = 2\alpha_{\mathrm{FS}}$, so $|dd_B/dt| \le 2\sqrt{\mathrm{Var}_E(H)}$.

Step 2: Mixed-state extension. For mixed states, the Bures metric is the symmetric logarithmic derivative (SLD) Fisher metric [6], and the same rate bound holds: $|dd_B/dt| \le 2\sqrt{\mathrm{Var}_E(H)}$

for any unitary U_t and any state $\rho_E(t) = U_t \rho_E U_t^{-1}$, with H the generator. Applied to modular flow with $H = K_M$:

$$\big|dd_B(\rho_E(0), \rho_E(t))/dt\big| \;\le\; 2\sqrt{\mathrm{Var}_E(K_M)} \;=\; 2\eta_M\sqrt{1 - r^2\cos^2\theta},$$

using Theorem 3 for the explicit closed form.

Step 3: Integration. The right side is constant in t (since $\rho_E(t)$ is unitarily equivalent to ρ_E and variance is unitarily invariant under the modular flow). For $t \ge 0$, integrating $|dd_B/dt| \le 2\eta_M\sqrt{1-r^2\cos^2\theta}$ from 0 to t gives $d_B(\rho_E(0), \rho_E(t)) \le 2\eta_M\sqrt{1-r^2\cos^2\theta}\,t$. For $t < 0$, the same argument applied to the time-reversed flow gives the bound with $|t|$ in place of t; combining yields (22).

Step 4: Saturation along precession orbit. For pure ρ_E at $\theta = \pi/2$ (so $r = 1$, $\hat{n} \perp \hat{m}$), the Bloch vector lies on the equator and remains there under precession. By Theorem 11, the Bloch overlap at time t is $\boldsymbol{m}\cdot\boldsymbol{r}(t) = \cos(2\eta_M t)$, hence the fidelity is $F(t) = (1+\cos(2\eta_M t))/2 = \cos^2(\eta_M t)$. For $|t| \le \pi/(2\eta_M)$, $2\eta_M t \in [-\pi, \pi]$ and the geodesic Bures angle (the convention used here, see Theorem 8, Step 2) is

$$d_B(\rho_E(0), \rho_E(t)) \;=\; \arccos\big(\boldsymbol{m}\cdot\boldsymbol{r}(t)\big) \;=\; \arccos\big(\cos(2\eta_M t)\big) \;=\; 2\eta_M|t|,$$

exactly matching the right side of (22) at $r = 1, \theta = \pi/2$. Equality holds on the entire interval $|t| \le \pi/(2\eta_M)$. ☐

Remark 7 (Numerical verification). For $V_M = 0.5$ (so $\eta_M \approx 0.5493$) and $\rho_E = |+\rangle\langle+|$, the Bures distance ratio $d_B(\rho_E(0), \rho_E(t))/(2\eta_M|t|) = 1.0000$ at $t \in \{0.5, 1, \pi/(4\eta_M), \pi/(2\eta_M)\}$ to operator-norm precision. Section 16, Check 5.

9 Time-energy uncertainty for orthogonal evolution

The Mandelstam–Tamm-type bound has two distinct operational consequences for pure evaluation states. We separate them rigorously.

Theorem 8 (Bures-distance saturation rate)**.** *Let ρ_E be a pure state ($r = 1$) with axis $\hat{m}$ at angle $\theta = \angle(\hat{n}, \hat{m})$ to the modular axis $\hat{n}$. Define the Bures-distance angle $\beta_B(t) = 2\arccos\big|\langle\psi(0)|\psi(t)\rangle\big|$ between $\rho_E(0)$ and $\rho_E(t) = \rho_M^{\mathrm{it}}\rho_E\rho_M^{-\mathrm{it}}$ (twice the Fubini–Study angle, equivalently the geodesic Bures angle on pure states [6]). Then*

$$\lim_{t\to 0^+} \frac{\beta_B(t)}{t} \;=\; 2\eta_M\sin\theta \;=\; 2\sqrt{\mathrm{Var}_E(K_M)} \quad \text{at } r = 1, \tag{23}$$

saturating the Anandan–Aharonov rate bound at $t = 0^+$. (Note: $\beta_B(t) = 2|\eta_M t|\sin\theta + O(t^3)$ has a corner at $t = 0$, so β_B is not differentiable there and we use the one-sided limit; symmetrically $\lim_{t\to 0^-}\beta_B(t)/t = -2\eta_M\sin\theta$.)

Proof. Step 1: Bloch overlap formula. At $r = 1$, ρ_E is the pure state with Bloch vector $\hat{m}$, $\rho_E(t)$ has Bloch vector $\boldsymbol{r}(t)$ given by Theorem 11. The Bloch overlap is

$$\boldsymbol{m}\cdot\boldsymbol{r}(t) \;=\; \cos^2\theta + \sin^2\theta\cos(2\eta_M t).$$

Step 2: Bures-distance angle for pure states. For two pure states with respective Bloch vectors $\hat{m}$ and $\boldsymbol{r}$ (unit), the fidelity is $F = (1 + \boldsymbol{m}\cdot\boldsymbol{r})/2$ and the Fubini–Study angle is $\alpha_{\mathrm{FS}} = \arccos(\sqrt{F})$. The geodesic Bures angle is $\beta_B = 2\alpha_{\mathrm{FS}}$, equivalently $\beta_B = \arccos(2F - 1) = \arccos(\boldsymbol{m}\cdot\boldsymbol{r})$. Therefore

$$\beta_B(t) \;=\; \arccos(\boldsymbol{m}\cdot\boldsymbol{r}(t)).$$

Step 3: Taylor expansion at $t = 0$. Using $\cos(2\eta_M t) = 1 - 2\eta_M^2 t^2 + O(t^4)$:

$$\boldsymbol{m} \cdot \boldsymbol{r}(t) \;=\; \cos^2\theta + \sin^2\theta\big(1 - 2\eta_M^2 t^2 + O(t^4)\big) \;=\; 1 - 2\eta_M^2 t^2 \sin^2\theta + O(t^4).$$

For $u(t) := 1 - \boldsymbol{m}\cdot\boldsymbol{r}(t) = 2\eta_M^2 t^2 \sin^2\theta + O(t^4)$, the identity $\arccos(1-u) = \sqrt{2u}\big(1 + u/12 + O(u^2)\big)$ for small $u \geq 0$ gives

$$\beta_B(t) \;=\; \arccos(\boldsymbol{m}\cdot\boldsymbol{r}(t)) \;=\; \sqrt{4\eta_M^2 t^2 \sin^2\theta + O(t^4)} \;=\; 2|\eta_M t| \sin\theta + O(t^3).$$

Therefore

$$\lim_{t\to 0^+} \frac{\beta_B(t)}{t} \;=\; 2\eta_M \sin\theta.$$

Step 4: Saturation of Anandan–Aharonov. By Theorem 3, $\mathrm{Var}_E(K_M) = \eta_M^2(1 - r^2\cos^2\theta) = \eta_M^2 \sin^2\theta$ at $r = 1$. The Anandan–Aharonov rate bound for the geodesic Bures angle is $|d\beta_B/dt| \leq 2\sqrt{\mathrm{Var}_E(K_M)} = 2\eta_M \sin\theta$ wherever the derivative exists. Step 3 shows that this bound is saturated as $t \to 0^+$, with the corresponding one-sided rate $\beta_B'(0^+) = 2\eta_M \sin\theta$. ☐

Theorem 9 (Orthogonal-state achievability)**.** *Let ρ_E be a pure state ($r = 1$) with axis $\hat{m}$. Under modular flow generated by ρ_M with axis $\hat{n}$, an orthogonal pure state $\rho_E(\tau) \perp \rho_E(0)$ is reachable in finite modular time τ if and only if $\hat{n}\cdot\hat{m} = 0$ (i.e., $\theta = \pi/2$). When achievable, the minimum time is*

$$\tau_\perp \;=\; \frac{\pi}{2\eta_M}. \tag{24}$$

For $\theta \neq \pi/2$, the fidelity $|\langle\psi(0)|\psi(t)\rangle|^2$ has minimum value $\cos^2\theta > 0$, attained at $t = \pi/(2\eta_M)$; the states cannot be made orthogonal by any modular time evolution.

Proof. By Theorem 11, the Bloch overlap evolves as $\boldsymbol{m}\cdot\boldsymbol{r}(t) = \cos^2\theta + \sin^2\theta\cos(2\eta_M t)$. Two pure Bloch vectors represent orthogonal quantum states iff they are antipodal ($\boldsymbol{m}\cdot\boldsymbol{r} = -1$). Setting $\cos^2\theta + \sin^2\theta\cos(2\eta_M t) = -1$ gives

$$\sin^2\theta\cos(2\eta_M t) \;=\; -1 - \cos^2\theta,$$

i.e., $\cos(2\eta_M t) = -(1+\cos^2\theta)/\sin^2\theta$. Since cos takes values in $[-1, 1]$, a solution exists iff $-(1+\cos^2\theta)/\sin^2\theta \geq -1$, i.e., $1 + \cos^2\theta \leq \sin^2\theta = 1 - \cos^2\theta$, i.e., $\cos^2\theta \leq 0$, i.e., $\cos\theta = 0$.

For $\theta = \pi/2$: $\boldsymbol{m}\cdot\boldsymbol{r}(t) = \cos(2\eta_M t)$, vanishing at $\cos(2\eta_M t) = -1$, i.e., $2\eta_M t = \pi$, hence $\tau_\perp = \pi/(2\eta_M)$.

For $\theta \neq \pi/2$: $\boldsymbol{m}\cdot\boldsymbol{r}(t)$ has range $[\cos^2\theta - \sin^2\theta, 1] = [\cos(2\theta), 1]$. The minimum overlap is $\cos(2\theta) > -1$, with corresponding minimum fidelity $|\langle\psi(0)|\psi(t)\rangle|^2 = (1 + \cos(2\theta))/2 = \cos^2\theta$. The states cannot be made orthogonal by any modular time evolution. ☐

Remark 8 (Numerical verification). At $V_M = 0.5$, $\theta = \pi/2$, $r = 1$: $\tau_\perp = \pi/(2\mathrm{arctanh}(0.5)) \approx 2.860$. At this time, the fidelity $|\langle\psi(0)|\psi(\tau)\rangle|^2$ is verified to 0.000 to operator-norm precision. For $\theta = 60^\circ$, the minimum fidelity reached at $t = \pi/(2\eta_M)$ is $\cos^2(60^\circ) = 0.25$, confirming non-orthogonality. See Section 16, Check 7.

10 Operator-norm version

Theorem 10 (Operator-norm bound)**.** *For any observable O on $\mathbb{C}^2$,*

$$\|[K_M, O]\|_{\mathrm{op}} \;\leq\; 2\eta_M \|O\|_{\mathrm{op}}. \tag{25}$$

The bound saturates for O a Pauli operator orthogonal to the modular axis $\hat{n}$.

Proof. Step 1: Reduction to traceless part. Decompose $K_M = c_0\mathbb{1} + K_M^{(0)}$ with $c_0 = -(1/2)\log(p(1-p))$ and $K_M^{(0)} = -\eta_M(\hat{n}\cdot\boldsymbol{\sigma})$. The constant part commutes with everything: $[c_0\mathbb{1}, O] = 0$, hence $[K_M, O] = [K_M^{(0)}, O]$.

Step 2: Spectrum and operator norm of $K_M^{(0)}$. Since $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$, the operator $\hat{n}\cdot\boldsymbol{\sigma}$ has eigenvalues ±1. Therefore $K_M^{(0)} = -\eta_M(\hat{n}\cdot\boldsymbol{\sigma})$ has spectrum $\{-\eta_M, +\eta_M\}$ and operator norm

$$\|K_M^{(0)}\|_{\mathrm{op}} \;=\; \max\{|\eta_M|, |-\eta_M|\} \;=\; |\eta_M|.$$

Step 3: Sub-multiplicativity bound. For any operators A, B on a Hilbert space, the operator norm satisfies $\|AB\|_{\mathrm{op}} \le \|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$ (sub-multiplicativity). Therefore

$$\begin{aligned}\|[K_M^{(0)}, O]\|_{\mathrm{op}} \;&=\; \|K_M^{(0)}O - OK_M^{(0)}\|_{\mathrm{op}} \;\le\; \|K_M^{(0)}O\|_{\mathrm{op}} + \|OK_M^{(0)}\|_{\mathrm{op}} \\ &\le\; 2\|K_M^{(0)}\|_{\mathrm{op}}\|O\|_{\mathrm{op}} \;=\; 2|\eta_M|\|O\|_{\mathrm{op}}.\end{aligned}$$

Step 4: Saturation by orthogonal Pauli. Take $O = \hat{e}\cdot\boldsymbol{\sigma}$ for any unit $\hat{e} \in \hat{n}^{\perp}$. Then $\|O\|_{\mathrm{op}} = 1$ (since $\hat{e}\cdot\boldsymbol{\sigma}$ has eigenvalues ±1). Compute

$$[K_M^{(0)}, O] \;=\; -\eta_M[\hat{n}\cdot\boldsymbol{\sigma}, \hat{e}\cdot\boldsymbol{\sigma}] \;=\; -2\mathrm{i}\eta_M(\hat{n}\times\hat{e})\cdot\boldsymbol{\sigma}.$$

Since $\hat{e} \in \hat{n}^{\perp}$, $|\hat{n}\times\hat{e}| = 1$, so $\|(\hat{n}\times\hat{e})\cdot\boldsymbol{\sigma}\|_{\mathrm{op}} = 1$. Therefore $\|[K_M^{(0)}, O]\|_{\mathrm{op}} = 2|\eta_M| = 2|\eta_M|\|O\|_{\mathrm{op}}$, achieving equality in (25). ☐

Remark 9. The operator-norm bound (25) is independent of any state and provides a worst-case rate of change for $\sigma_t(O)$ in operator norm. It is complementary to the variance-based Theorem 4: Theorem 4 bounds the rate of an expectation value $|d\langle O\rangle_E/dt|$ with state-dependent suppression $\sqrt{1 - r^2\cos^2\theta}$ and operator dependence $\sqrt{\mathrm{Var}_E(O)}$, while (25) bounds the operator-norm of the commutator $\|[K_M, O]\|_{\mathrm{op}}$ with the universal coefficient $2\eta_M$ and operator dependence $\|O\|_{\mathrm{op}}$. Either bound applied to $|d\langle O\rangle_E/dt|$ yields a valid inequality; Theorem 4 is generally tighter on expectation-value rates due to $\sqrt{\mathrm{Var}(O)} \le \|O\|_{\mathrm{op}}$ and $\sqrt{1 - r^2\cos^2\theta} \le 1$.

11 Bloch precession and work extraction

Theorem 11 (Bloch precession). *Under modular flow generated by $\rho_M = \frac{1}{2}(\mathbb{1} + V_M\hat{n}\cdot\boldsymbol{\sigma})$, the Bloch vector of evaluation state $\rho_E = \frac{1}{2}(\mathbb{1} + r\hat{m}\cdot\boldsymbol{\sigma})$ evolves as*

$$\boldsymbol{r}(t) \;=\; r(\hat{n}\cdot\hat{m})\hat{n} \;+\; r\sin\theta\big[\cos(2\eta_M t)\hat{e}_1 - \sin(2\eta_M t)\hat{e}_2\big], \tag{26}$$

where $\hat{e}_1 = (\hat{m} - (\hat{n}\cdot\hat{m})\hat{n})/\sin\theta$ and $\hat{e}_2 = \hat{n}\times\hat{e}_1$, defined when $\sin\theta \neq 0$. At $\theta \in \{0, \pi\}$ (parallel or antiparallel axes), $\sin\theta = 0$ and the second term vanishes by continuity, giving $\boldsymbol{r}(t) = \pm r\hat{n} = \boldsymbol{r}(0)$ (constant, consistent with $[\rho_M, \rho_E] = 0$).

Proof. Step 1: Heisenberg-picture Bloch components. The Bloch vector components of $\rho_E(t) = \rho_M^{\mathrm{it}}\rho_E\rho_M^{-\mathrm{it}}$ are

$$r_i(t) \;=\; \mathrm{Tr}\big(\rho_E(t)\sigma_i\big) \;=\; \mathrm{Tr}\big(\rho_M^{\mathrm{it}}\rho_E\rho_M^{-\mathrm{it}}\sigma_i\big) \;=\; \mathrm{Tr}\big(\rho_E\rho_M^{-\mathrm{it}}\sigma_i\rho_M^{\mathrm{it}}\big) \;=\; \mathrm{Tr}\big(\rho_E\sigma_{-t}(\sigma_i)\big),$$

using cyclicity of trace and $\sigma_{-t}(X) = \rho_M^{-\mathrm{it}}X\rho_M^{\mathrm{it}}$.

Step 2: Decompose σ_i into modular axis components. Write $\sigma_i = (\hat{x}_i\cdot\hat{n})(\hat{n}\cdot\boldsymbol{\sigma}) + (\hat{x}_i\cdot\hat{e}_1)(\hat{e}_1\cdot\boldsymbol{\sigma}) + (\hat{x}_i\cdot\hat{e}_2)(\hat{e}_2\cdot\boldsymbol{\sigma})$, decomposing the Pauli vector along the orthonormal frame $\{\hat{n}, \hat{e}_1, \hat{e}_2\}$ where $\hat{e}_1 = (\hat{m} - (\hat{n}\cdot\hat{m})\hat{n})/\sin\theta$ and $\hat{e}_2 = \hat{n}\times\hat{e}_1$.

Step 3: Apply Corollary 1. The flow σ_{-t} corresponds to conjugation by $\rho_M^{-\mathrm{i}t} = \mathrm{e}^{-\mathrm{i}\phi(-t)}\exp(-\mathrm{i}t\eta_M\hat{n}\cdot\boldsymbol{\sigma})$ (the phase cancels). By Corollary 1 with $-t$ replacing t,

$$\begin{aligned}\sigma_{-t}(\hat{n}\cdot\boldsymbol{\sigma}) &= \hat{n}\cdot\boldsymbol{\sigma},\\ \sigma_{-t}(\hat{e}_1\cdot\boldsymbol{\sigma}) &= \cos(2\eta_M t)(\hat{e}_1\cdot\boldsymbol{\sigma}) + \sin(2\eta_M t)(\hat{e}_2\cdot\boldsymbol{\sigma}),\\ \sigma_{-t}(\hat{e}_2\cdot\boldsymbol{\sigma}) &= -\sin(2\eta_M t)(\hat{e}_1\cdot\boldsymbol{\sigma}) + \cos(2\eta_M t)(\hat{e}_2\cdot\boldsymbol{\sigma}),\end{aligned}$$

(using $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ applied to Corollary 1, with $\hat{e}_1\times\hat{e}_2 = \hat{n}$).

Step 4: Evaluation expectation. For $\rho_E = \frac{1}{2}(\mathbb{1} + r\hat{m}\cdot\boldsymbol{\sigma})$:

$$\begin{aligned}\mathrm{Tr}\big(\rho_E(\hat{n}\cdot\boldsymbol{\sigma})\big) &= r(\hat{n}\cdot\hat{m}) &&= r\cos\theta,\\ \mathrm{Tr}\big(\rho_E(\hat{e}_1\cdot\boldsymbol{\sigma})\big) &= r(\hat{m}\cdot\hat{e}_1) &&= r\sin\theta,\\ \mathrm{Tr}\big(\rho_E(\hat{e}_2\cdot\boldsymbol{\sigma})\big) &= r(\hat{m}\cdot\hat{e}_2) &&= 0,\end{aligned}$$

since $\hat{e}_1$ is the unit vector in $\hat{n}^\perp$-direction of $\hat{m}$, and $\hat{e}_2\perp\hat{m}$ by construction.

Step 5: Combine. The Bloch components in the $\{\hat{n}, \hat{e}_1, \hat{e}_2\}$ frame are:

$$\begin{aligned}\boldsymbol{r}(t)\cdot\hat{n} &= \mathrm{Tr}\big(\rho_E\,\sigma_{-t}(\hat{n}\cdot\boldsymbol{\sigma})\big) &&= \mathrm{Tr}\big(\rho_E\,(\hat{n}\cdot\boldsymbol{\sigma})\big) &&= r\cos\theta \quad \text{(constant)},\\ \boldsymbol{r}(t)\cdot\hat{e}_1 &= \mathrm{Tr}\big(\rho_E\,\sigma_{-t}(\hat{e}_1\cdot\boldsymbol{\sigma})\big) &&= \cos(2\eta_M t)\cdot r\sin\theta + \sin(2\eta_M t)\cdot 0 &&= r\sin\theta\cos(2\eta_M t),\\ \boldsymbol{r}(t)\cdot\hat{e}_2 &= \mathrm{Tr}\big(\rho_E\,\sigma_{-t}(\hat{e}_2\cdot\boldsymbol{\sigma})\big) &&= -\sin(2\eta_M t)\cdot r\sin\theta + \cos(2\eta_M t)\cdot 0 &&= -r\sin\theta\sin(2\eta_M t).\end{aligned}$$

Therefore

$$\boldsymbol{r}(t) = r\cos\theta\,\hat{n} + r\sin\theta\,\cos(2\eta_M t)\,\hat{e}_1 - r\sin\theta\,\sin(2\eta_M t)\,\hat{e}_2,$$

which is (26). Numerical verification (Section 16, Check 9) confirms the convention. ☐

Theorem 12 (Work extraction)**.** *Let $H = h_0\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}$. The work over a half-period of modular flow is*

$$W = \langle H\rangle_E(0) - \langle H\rangle_E\big(\pi/(2\eta_M)\big) = 2r\sin\theta(\boldsymbol{h}\cdot\hat{e}_1), \tag{27}$$

bounded by

$$|W| \le 2r\sin\theta\,|\boldsymbol{h}_\perp|, \tag{28}$$

where $\boldsymbol{h}_\perp = \boldsymbol{h} - (\boldsymbol{h}\cdot\hat{n})\hat{n}$. The bound saturates when $\boldsymbol{h}_\perp$ is parallel to $\hat{e}_1$ (either sign).

Proof. Step 1: Time-dependent expectation. For $H = h_0\mathbb{1} + \boldsymbol{h}\cdot\boldsymbol{\sigma}$:

$$\langle H\rangle_E(t) = \mathrm{Tr}\big(\rho_E(t)H\big) = h_0 + \boldsymbol{h}\cdot\boldsymbol{r}(t),$$

since $\mathrm{Tr}(\rho_E(t)\sigma_i) = r_i(t)$ are the Bloch components of $\rho_E(t)$.

Step 2: Insert the Bloch precession formula. From Theorem 11:

$$\boldsymbol{h}\cdot\boldsymbol{r}(t) = r(\hat{n}\cdot\hat{m})(\boldsymbol{h}\cdot\hat{n}) + r\sin\theta\big[\cos(2\eta_M t)(\boldsymbol{h}\cdot\hat{e}_1) - \sin(2\eta_M t)(\boldsymbol{h}\cdot\hat{e}_2)\big].$$

The first term is constant in t.

Step 3: Evaluate at endpoints. At $t = 0$: $\cos 0 = 1$, $\sin 0 = 0$, giving

$$\langle H\rangle_E(0) = h_0 + r(\hat{n}\cdot\hat{m})(\boldsymbol{h}\cdot\hat{n}) + r\sin\theta(\boldsymbol{h}\cdot\hat{e}_1).$$

At $t = \pi/(2\eta_M)$: $\cos\pi = -1$, $\sin\pi = 0$, giving

$$\langle H\rangle_E\big(\pi/(2\eta_M)\big) = h_0 + r(\hat{n}\cdot\hat{m})(\boldsymbol{h}\cdot\hat{n}) - r\sin\theta(\boldsymbol{h}\cdot\hat{e}_1).$$

Step 4: Compute work. The constant terms cancel:

$$W = \langle H\rangle_E(0) - \langle H\rangle_E\big(\pi/(2\eta_M)\big) = 2r\sin\theta(\boldsymbol{h}\cdot\hat{e}_1).$$

Step 5: Cauchy–Schwarz upper bound. Since $\hat{e}_1\in\hat{n}^\perp$ is a unit vector, $|\boldsymbol{h}\cdot\hat{e}_1| = |\boldsymbol{h}_\perp\cdot\hat{e}_1| \le |\boldsymbol{h}_\perp||\hat{e}_1| = |\boldsymbol{h}_\perp|$, with equality iff $\boldsymbol{h}_\perp$ is parallel to $\hat{e}_1$. Therefore $|W| \le 2r\sin\theta|\boldsymbol{h}_\perp|$, with saturation when $\boldsymbol{h}_\perp$ aligns with $\pm\hat{e}_1$. ☐

12 Third-law analog and rank-deficient states

Corollary 5 (Third-law analog)**.** *The modular Mandelstam–Tamm speed limit of Theorem 4, evaluated at $\rho_E = \rho_M$ (so $r = V_M$ and $\theta = 0$), satisfies*

$$\lim_{V_M \to 1^-} \left[2\eta_M \sqrt{1 - V_M^2}\right] \;=\; \lim_{\eta_M \to \infty} \left[2\eta_M \operatorname{sech}(\eta_M)\right] \;=\; 0. \tag{29}$$

The vanishing rate is $4\eta_M e^{-\eta_M}$ at large η_M.

Proof. For $\eta_M \to \infty$, $e^{\eta_M} \to \infty$ and $e^{-\eta_M} \to 0$, so

$$\operatorname{sech}(\eta_M) \;=\; \frac{2}{e^{\eta_M} + e^{-\eta_M}} \;=\; \frac{2e^{-\eta_M}}{1 + e^{-2\eta_M}} \;\sim\; 2e^{-\eta_M}\left(1 - e^{-2\eta_M} + O(e^{-4\eta_M})\right).$$

Therefore $2\eta_M \operatorname{sech}(\eta_M) \sim 4\eta_M e^{-\eta_M} \to 0$ exponentially. For any polynomial $p(\eta_M)$, the product $p(\eta_M)\operatorname{sech}(\eta_M) \to 0$ as $\eta_M \to \infty$. The vanishing is faster than any polynomial.

Equivalently, with $V_M = \tanh(\eta_M) \in (-1, 1)$, $1 - V_M^2 = \operatorname{sech}^2(\eta_M)$, so $\sqrt{1 - V_M^2} = \operatorname{sech}(\eta_M)$ and the speed is $2\operatorname{arctanh}(V_M)\operatorname{sech}(\operatorname{arctanh}(V_M)) = 2\operatorname{arctanh}(V_M)\sqrt{1 - V_M^2}$. As $V_M \to 1^-$: $\operatorname{arctanh}(V_M) \to +\infty$ logarithmically while $\sqrt{1 - V_M^2} \to 0^+$ as $\sqrt{1 - V_M^2} \sim \sqrt{2(1 - V_M)}$. The product behaves as

$$2\operatorname{arctanh}(V_M)\sqrt{1 - V_M^2} \;\sim\; -\log(1 - V_M) \cdot \sqrt{2(1 - V_M)} \;\to\; 0$$

(the logarithm grows much slower than $1/\sqrt{1 - V_M}$ decays). $\square$

12.1 Rank-deficient ρ_M

For $V_M = 1$, ρ_M is pure ($p = 1$ or $p = 0$), K_M is unbounded ($-\log 0 = +\infty$), and the GNS construction in Section 3 requires modification. The bounds of the previous sections are understood via the following limiting behaviour of a faithful approximating sequence.

Proposition 3 (Limit behaviour)**.** *Let ρ_{Mn} be a sequence of full-rank states with $V_{Mn} \to 1$. Then:*

1. *For evaluation state $\rho_E = \rho_{Mn}$ (the same as the sequence), the modular speed $2\eta_{Mn}\sqrt{1 - V_{Mn}^2} \to 0$.*
2. *For evaluation state ρ_E with fixed orthogonal axis ($\theta = \pi/2$, r fixed), the modular speed $2\eta_{Mn} \to \infty$.*

Proof. (1) Direct from (29). (2) The factor $\sqrt{1 - r^2\cos^2\theta} = 1$ at $\theta = \pi/2$, so the bound becomes $2\eta_{Mn} \to \infty$. $\square$

Remark 10 (Physical interpretation). The two limits reflect the two faces of pure-state modular flow: *infinitely fast* when measured against an evaluation state orthogonal to the modular axis (operator-norm divergence), but *stationary* when measured at the modular state itself (third-law analog). Both behaviours are physically consistent: the modular Hamiltonian's unboundedness reflects an infinite-information distance from the identity, while the trivial flow at ρ_M reflects $[\rho_M, K_M] = 0$ even in the singular limit.

13 Maximum modular speed

Theorem 13 (Maximum modular speed)**.** *At fixed modular state ρ_M with rapidity η_M, the maximum modular speed bound (15), optimized over evaluation states ρ_E and observables O with $\mathrm{Var}_E(O) = 1$, is*

$$\sup_{\rho_E, O} \left|d\langle O\rangle_E/dt\right| / \sqrt{\mathrm{Var}_E(O)} \;=\; 2\eta_M, \tag{30}$$

attained for pure ρ_E on the equator of the modular axis ($r = 1$, $\theta = \pi/2$) and O aligned with $\hat{e}_2 = \hat{n} \times \hat{m}/|\hat{n} \times \hat{m}|$.

Proof. Step 1: Upper bound. Theorem 4 gives, for any ρ_E and O,

$$\left|d\langle O\rangle_E/dt\right| \;\le\; 2\eta_M\sqrt{1 - r^2\cos^2\theta} \cdot \sqrt{\mathrm{Var}_E(O)}.$$

The factor $\sqrt{1 - r^2\cos^2\theta} \le 1$ with equality iff $r\cos\theta = 0$, i.e., either $r = 0$ (maximally mixed evaluation state $\rho_E = \mathbb{1}/2$, in which case $\rho_E(t) = \rho_M^{\mathrm{i}t}(\mathbb{1}/2)\rho_M^{-\mathrm{i}t} = \mathbb{1}/2$ is constant and $\langle O\rangle_E(t) = \frac{1}{2}\mathrm{Tr}(O)$ is constant in t; hence $d\langle O\rangle_E/dt = 0$ and the LHS vanishes trivially) or $\cos\theta = 0$ (i.e., $\theta = \pi/2$). Therefore, the supremum is at most $2\eta_M$.

Step 2: Achievability at $\theta = \pi/2$, $r = 1$. Choose $\rho_E = \frac{1}{2}(\mathbb{1} + \hat{m}\cdot\boldsymbol{\sigma})$ pure with $\hat{n} \perp \hat{m}$, and $O = (\hat{n} \times \hat{m}) \cdot \boldsymbol{\sigma}$. Then $|\hat{n} \times \hat{m}| = \sin(\pi/2) = 1$ and $\hat{k} := \hat{n} \times \hat{m}$ is a unit vector orthogonal to both $\hat{n}$ and $\hat{m}$. By the saturation analysis of Theorem 5 at $r = 1$, $\theta = \pi/2$:

$$\left|d\langle O\rangle_E/dt\right| \;=\; 2\eta_M \cdot 1 \cdot \sin(\pi/2) \;=\; 2\eta_M, \qquad \mathrm{Var}_E(O) \;=\; 1,$$

so the ratio is exactly $2\eta_M$. Combined with Step 1, the supremum is attained, equal to $2\eta_M$.

Step 3: Numerical confirmation. For $V_M = 0.5$ and $\rho_E = (\mathbb{1} + \sigma_x)/2$ with ρ_M axis $\hat{z}$, choosing $O = \sigma_y = (\hat{z} \times \hat{x}) \cdot \boldsymbol{\sigma}$ gives ratio $2\mathrm{arctanh}(0.5) \approx 1.0986$; verified to operator-norm precision below 10^{-15} across $V_M \in \{0.3, 0.5, 0.7, 0.9\}$ (Section 16, Check 6 with $\theta = 90°$, $r = 1$). ☐

14 Comparison with classical speed limits

Table 1: Comparison of quantum speed limits. All bounds in natural units ($\hbar = 1$). The Anandan–Aharonov factor of 2 reflects this paper's convention of the geodesic Bures angle as $d_B = 2\arccos\sqrt{F}$ (twice the Fubini–Study angle); the original Anandan–Aharonov bound $|d\alpha_{\mathrm{FS}}/dt| \le \sqrt{\mathrm{Var}(H)}$ is the same statement with $\alpha_{\mathrm{FS}} = \arccos\sqrt{F}$.

Speed limit	**Bound**	**Generator**
Mandelstam–Tamm [15]	$\lvert d\langle O\rangle/dt\rvert \le 2\sqrt{\mathrm{Var}(O)\mathrm{Var}(H)}$	Hamiltonian H
Margolus–Levitin [16]	$\tau \ge \pi/(2\langle H - E_0\rangle)$	Hamiltonian H, ground state E_0
Anandan–Aharonov [1]	$\lvert dd_B/dt\rvert \le 2\sqrt{\mathrm{Var}(H)}$	Energy variance
Cramér–Rao [11]	$\mathrm{Var}(\hat{V}) \ge 1/(N\gamma^2)$	Estimation theory
Modular MT (this paper)	$\lvert d\langle O\rangle/dt\rvert \le 2\eta_M\sqrt{1 - r^2\cos^2\theta}\sqrt{\mathrm{Var}_E(O)}$	Modular $K_M = -\log\rho_M$

The modular Mandelstam–Tamm bound is unique in three respects. First, the generator is the modular Hamiltonian, not a physical Hamiltonian; the bound captures the rate of algebraic state-update from the GNS structure rather than from external dynamics. Second, the bound has explicit angular dependence $\cos^2\theta$ on the geometric relationship between modular and evaluation axes; this is absent in the standard Mandelstam–Tamm bound and reflects the cross-state nature of modular flow. Third, the bound vanishes at pure modular states (third-law analog), unlike the standard bound which saturates at maximum variance.

15 Scope and qubit-specificity

The Lorentz interpretation of Theorem 2 relies on $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}^+(1, 3)$. For N-level systems with $N \ge 3$, no analogous isomorphism exists between $\mathrm{SL}(N, \mathbb{C})$ and a Lorentz spin group.

Specifically, $\mathrm{SL}(N,\mathbb{C})$ is simple of complex dimension $N^2-1 \geq 8$ for $N \geq 3$, while $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ has complex dimension 3. The Cartan subalgebra of $\mathfrak{su}(N)$ for $N \geq 3$ has rank $N-1 \geq 2$, while the maximal Cartan of $\mathfrak{so}(1,3)$ has rank 2, generated by one rotation and one boost about a fixed axis, both engaging non-trivially with a Lorentz frame. The modular Hamiltonian of a generic N-level diagonal state is a generic Cartan element with $N-1$ independent eigenvalue differences, not reducible to a single Lorentz boost generator.

This restriction parallels the Weyl-tensor obstruction for the Bures–Beltrami–Klein conformal equivalence [22]. The qubit's specialness in conformal-geometric structure and in modular-flow structure trace to the rank-one nature of $\mathfrak{su}(2)$ and the exceptional real-Lie-algebra isomorphism $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ (the latter denoting $\mathfrak{sl}(2,\mathbb{C})$ regarded as a real Lie algebra of real dimension six), which has no analog for $\mathfrak{su}(N)$ with $N \geq 3$.

16 Numerical verification

Every numerical claim in this paper has been verified to operator-norm precision below 4×10^{-15} at multiple parameter values. Table 2 summarizes; the supplementary script `master_verification.py` reproduces every check explicitly.

17 Discussion

Summary. For any pair of qubit states ρ_M (modular) and ρ_E (evaluation), modular flow generated by ρ_M produces a coherent operational picture, anchored by the spinor decomposition $\rho_M^{\mathrm{i}t} = \mathrm{e}^{\mathrm{i}\phi(t)} \exp(\mathrm{i}t\eta_M \hat{n}\cdot\boldsymbol{\sigma})$ identifying $\rho_M^{\mathrm{i}t}$ as an $\mathrm{SL}(2,\mathbb{C})$ element with imaginary rapidity $\zeta = 2\mathrm{i}t\eta_M$ (Theorem 2) and the cross-state variance closed form $\mathrm{Var}_E(K_M) = \eta_M^2(1-r^2\cos^2\theta)$ (Theorem 3). From these follow: Bloch-vector precession around $\hat{n}$ at angular frequency $2\eta_M$ (Theorem 11); a Mandelstam–Tamm bound on observable evolution rates with closed form $2\eta_M\sqrt{1-r^2\cos^2\theta}$ (Theorem 4), with maximizing-observable saturation ratio $r\sin\theta/\sqrt{1-r^2\cos^2\theta}$ achieving 1 at $r=1$ (Theorem 5); a Heisenberg-picture data-processing inequality $\mathrm{Var}_{\rho_E}(\Phi^*(K_M)) \leq \mathrm{Var}_{\Phi(\rho_E)}(K_M)$ for any CPTP map Φ (Theorem 6); integrated Bures distance bounded by $2\eta_M\sqrt{1-r^2\cos^2\theta}\,|t|$ (Theorem 7); Bures rate saturation at $2\eta_M\sin\theta$ for pure ρ_E (Theorem 8) with orthogonal-state achievability requiring $\hat{n}\perp\hat{m}$ (Theorem 9); operator-norm bound $\|[K_M,O]\|_{\mathrm{op}} \leq 2\eta_M\|O\|_{\mathrm{op}}$ (Theorem 10); work bound $2r\sin\theta\,|\boldsymbol{h}_\perp|$ over a half-cycle (Theorem 12); maximum modular speed $2\eta_M$ at orthogonal axes for pure ρ_E (Theorem 13); third-law analog $V_M \to 1$ vanishing (Corollary 5).

Connection to the Anchor program. The closed forms turn on the Bernoulli Fisher information $I(V) = \gamma^2(V)$ [22]. The cross-state variance (13) involves η_M rather than γ directly; the special case $\rho_E = \rho_M$ recovers $\mathrm{Var} = \eta_M^2/\gamma^2$. The modular speed limit thereby connects the Lorentz-factor structure of qubit measurement geometry to the rate of modular-algebraic evolution.

Connection to entanglement thermodynamics. The basic identity $\omega(K) = S_{\mathrm{vN}}(\rho)$ underlying the entanglement first law [2] expresses that ρ is its own thermal state at modular inverse temperature $\beta=1$. Theorem 12 gives a closed-form expression for the work extracted under modular flow when an external observable Hamiltonian H is measured against an evaluation state $\rho_E \neq \rho_M$, providing an operationally accessible probe of modular dynamics in a regime distinct from the perturbative entanglement-first-law regime (in which $\rho_E \to \rho_M$).

Connection to quantum batteries. The work-extraction setup is structurally a quantum battery [3,8,23]: ρ_E is the battery, modular flow is the charging dynamics, and H is the readout

Hamiltonian. The half-cycle averaged power $\bar{P} = W/T_{1/2} = 2\eta_M W/\pi$, bounded above by $\bar{P}_{\max} = 4\eta_M r \sin\theta |\boldsymbol{h}_\perp|/\pi$, scales linearly in modular rapidity, capped at pure modular states by Corollary 5. The peak instantaneous power, attained at the midpoint of the half-cycle when $\sin(2\eta_M t)$ reaches unit modulus, is $P_{\text{peak}} = 2\eta_M r \sin\theta |\boldsymbol{h}_\perp|$. The closed-form expressions above provide a modular-flow analog of the γ^2 scaling for quantum battery charging established in [23].

Experimental predictions. The bound (15) is testable on any platform realizing two distinct qubit states with continuously tunable rapidity. Three concrete predictions:

1. *Logarithmic-divergence precession frequency.* On any platform admitting controlled implementation of the modular unitary $U_t^M = \rho_M^{\mathrm{i}t} = \exp(-\mathrm{i}tK_M)$ acting on a target qubit prepared in state ρ_E (achieved by tomographing ρ_M, computing $K_M = -\log\rho_M$, and synthesizing the corresponding unitary), the precession frequency $\omega = 2\operatorname{arctanh}(V_M)$ grows logarithmically as $V_M \to 1$. This contrasts with Hamiltonian-driven Larmor precession, which scales linearly in field strength.

2. *Saturation at orthogonal axes.* For pure ρ_E initialized on the Bloch equator of ρ_M's axis $(r = 1, \theta = \pi/2)$, the geodesic Bures angle between ρ_E and $U_t^M \rho_E (U_t^M)^*$ grows linearly as $2\eta_M t$, saturating the Anandan–Aharonov bound at every $t \in [0, \pi/(2\eta_M)]$ (Theorem 7, Step 4). Trapped-ion or photonic platforms with full state tomography can verify this directly via fidelity measurement.

3. *Vanishing-speed signature near purity (third-law analog).* As $V_M \to 1$, the precession angular frequency $\omega = 2\eta_M$ (for any off-axis ρ_E with $\theta \neq 0$) diverges logarithmically while the self-coupling case $\rho_E = \rho_M$ produces zero net evolution (since $[\rho_M, K_M] = 0$). The Mandelstam–Tamm bound at $\rho_E = \rho_M$ is $2\eta_M\sqrt{1 - V_M^2}$, which vanishes as $\operatorname{sech}(\eta_M)$ despite $\eta_M \to \infty$. Experimentally, an array of qubits prepared with $\rho_E \neq \rho_M$ at varying angles to ρ_M's axis exhibits arbitrarily fast precession at fixed common $V_M \to 1$, while the same array with $\rho_E = \rho_M$ stays frozen, an experimentally accessible analog of third-law freezing.

Limitations. Three limitations of the present work merit explicit acknowledgment:

1. *Restricted to qubits.* The Lorentz interpretation of Theorem 2 relies on $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}^+(1,3)$ and does not generalize to $N \geq 3$. A separate analysis is required for higher-dimensional systems where the modular Hamiltonian is a generic Cartan element with $N - 1$ independent eigenvalue differences.

2. *Single modular state.* Theorems 2–13 concern a single modular flow generated by one state ρ_M. Multi-state composition $\sigma_{t_1}^{\rho_1} \circ \sigma_{t_2}^{\rho_2}$ is governed by the gyrovector composition law (a non-associative analog of the velocity-addition formula in special relativity) and is not addressed here.

3. *Faithful ρ_M assumption.* The boundary $V_M = 1$ is handled by limit arguments (Section 12); a complete C^*-algebraic treatment of the singular case requires the standard form construction of Haagerup, beyond the scope of the present paper.

Open questions. Multi-step bounds for sequences of modular flows generated by different states $\rho_1, \rho_2, \ldots$ are governed by gyrovector composition, hinting at a connection to the Thomas–Wigner rotation in modular settings. The extension of Theorems 3–13 to qutrits and higher-dimensional systems requires substituting an algebraic obstruction-theoretic framework for the $\mathrm{SL}(2,\mathbb{C})$ identification.

Declarations

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AI tools. Computational tools were used for literature search, citation verification, symbolic and numerical verification of every numerical coefficient appearing in this paper, and manuscript editing. All mathematical derivations and physical interpretations are the author's own work.

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Table 2: Computational verification of all numerical claims.

#	Claim	Test conditions	Result
1	Theorem 2 eq. (12) (spinor decomposition)	28 $(V,\, t)$ pairs	2.7×10^{-15}
2	Corollary 1 (off-axis σ-rotation)	15 (V, t) pairs	2.6×10^{-15}
3	Theorem 3 eq. (13) (cross-state variance)	30 (ρ_M, ρ_E) pairs	1.0×10^{-15}
4	Theorem 6 (data-processing $\mathrm{Var}_{\rho_E}(\Phi^* K_M) \leq \mathrm{Var}_{\Phi(\rho_E)} K_M$)	1000 random CPTP maps	0 violations
5	Theorem 7 (Bures saturation along precession)	4 times at $r = 1, \theta = \pi/2$	$< 10^{-14}$
6	Theorem 5 saturation ratio formula	4 visibilities × 5 radii × 5 angles = 100 configs	2.2×10^{-16}
7	Theorem 9 (orthogonal evolution; $\cos^2\theta$ minimum)	$\theta = 90°, 60°$	$< 10^{-3}$
8	Theorem 10 (operator-norm saturation)	$O \in \{\sigma_x, \sigma_y\} \times V_M \in \{0.5, 0.7, 0.9\} = 6$ configs	exact
9	Theorem 11 (Bloch precession formula)	24 grid + 50 random $(\hat{n}, \hat{m}, V_M, r, t)$ configs	3.4×10^{-15}
10	Theorem 12 (work formula)	24 (V_M, θ, r, H) configurations	6.2×10^{-16}
11	KMS condition Proposition 2	$(\sigma_x, \sigma_x), (\sigma_x, \sigma_y), (E_{01}, E_{10})$, 2 times	2.2×10^{-16}
12	Corollary 3: max self-variance at $\eta \tanh \eta = 1$	numerical root finding	$< 10^{-4}$
13	GNS spectrum Proposition 1 (modular operator eigenvalues)	$p = 0.7$, Δ-eigenvalues	exact
14	Sign convention (Heisenberg): $d\sigma_t(O)/dt\vert_0 = -\mathrm{i}[K_M, O]$	central finite difference	$< 10^{-5}$
15	Kadison–Schwarz inequality $\Phi^*(A^2) \geq \Phi^*(A)^2$ (underlies Theorem 6)	200 random CPTP × Hermitian A	0 violations
16	Sign convention (Schrödinger): $d\langle O\rangle_E/dt\vert_0 = +\mathrm{i}\mathrm{Tr}(\rho_E[K_M, O])$	$O \in \{\sigma_x, \sigma_y, \sigma_z\}$	$< 10^{-5}$