Fisher Information Is the Squared Lorentz Factor

Why the Qubit Is Special

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A perspective article establishing $I(V) = \gamma^2(V)$ as a structural bridge between quantum information geometry and relativistic kinematics, with the novel observation that the Bloch ball simultaneously supports spherical (QFI) and hyperbolic (Beltrami-Klein) Riemannian structures on the same underlying space.

Abstract

We show that the open qubit Bloch ball carries two canonical Riemannian structures that are exactly related by a simple conformal factor. Writing the Bures metric as $ds^2_{\text{Bures}} = \frac{dr^2}{4(1-r^2)} + \frac{r^2}{4}\, d\Omega^2$ and the Beltrami–Klein metric on hyperbolic space as $ds^2_{\text{BK}} = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{1-r^2}\, d\Omega^2$, we prove the conformal identity $ds^2_{\text{BK}} = 4(1-r^2)^{-1}\, ds^2_{\text{Bures}}$. The conformal factor $(1-r^2)^{-1}$ equals the squared Lorentz factor γ^2 and, for a binary visibility coordinate $V = 2p-1$, coincides with the Fisher information $I(V) = 1/(1-V^2)$.

We then show that this phenomenon is special to $N = 2$: for $N \geq 3$ the Bures metric on full-rank density matrices has nonconstant sectional curvature at the maximally mixed state, implying a nonvanishing Weyl tensor and forbidding conformal equivalence to any constant-curvature model. Finally, we outline an operational consequence in continuously monitored qubits, where the relevant state-update dynamics realize an $\mathrm{SL}(2,\mathbb{C})$ action and predict a Wigner/Thomas rotation for sequential non-collinear weak measurements.

Keywords: Fisher information, Bures metric, Bloch ball, Beltrami-Klein model, conformal equivalence, quantum estimation

1 Introduction

Gyrovector spaces, as Ungar [42] established, form the algebraic setting for analytic hyperbolic geometry, just as vector spaces form the algebraic setting for analytic Euclidean geometry. This paper reveals a new chapter in that program: the qubit Bloch ball—already known to carry gyrovector structure—simultaneously supports two canonical Riemannian metrics that are related by a conformal factor with deep physical meaning.

Consider a binary quantum measurement with outcome probability p. Define the *visibility* $V = 2p - 1 \in (-1, 1)$, so that $V = 0$ is a completely random outcome and $|V| = 1$ is a deterministic one. The Fisher information of this Bernoulli model, expressed in the visibility coordinate, is:

$$I(V) = \frac{1}{1 - V^2} = \gamma^2(V), \tag{1}$$

the squared Lorentz factor of special relativity.

This paper establishes that Eq. (1) is not a coincidence of functional form but reflects a deep structural connection between quantum measurement geometry and relativistic kinematics, grounded in the exceptional group isomorphism $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}^+(1, 3)$.

1.1 Scope and Claims

This paper's central results are: **the Bures metric and Beltrami-Klein metric on the qubit Bloch ball are conformally equivalent, with conformal factor $\gamma^2(r) = 1/(1 - r^2)$ equal to both the squared Lorentz factor and the Fisher information of a binary quantum measurement. This conformal structure is unique to qubits: for $N \geq 3$, the Weyl tensor of the Bures metric is nonvanishing, providing a curvature obstruction. Six complementary perspectives make the conformal factor physically meaningful.**

To our knowledge, the explicit tensor-level conformal identity $ds^2_{\mathrm{BK}} = 4\gamma^2\, ds^2_{\mathrm{Bures}}$ has not been stated in the existing Bures-geometry or gyrovector literatures, where related results appear at the level of either the Bures metric tensor (spherical/Uhlmann hemisphere) or hyperbolic/rapidity formulas for fidelity. Even though the algebraic conformal factor can be inferred once both metrics are written in Bloch coordinates, we have not found it stated explicitly in tensor-level form; our contribution is the packaging together with the sharp $N \geq 3$ obstruction and operational consequences. The paper makes no claims about quantum gravity, consciousness, or artificial intelligence.

Novelty statement. Building on the seminal identification by Chen and Ungar [8, 80, 81, 82] of the Bloch vector as a gyrovector in the Beltrami-Klein model, this paper extends their algebraic result to the level of Riemannian metrics and establishes the precise algebraic boundary of the phenomenon. Chen and Ungar's results operate at the scalar/distance-function level (expressing Bures fidelity through Einstein velocity addition); the present paper establishes the Riemannian tensor-level conformal equivalence, which is strictly stronger: it determines all angles, sectional curvatures, volumes, and geodesic structure simultaneously. While the individual mathematical ingredients are known—the Bernoulli Fisher information (textbook), $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{SO}^+(1, 3)$ (Penrose–Rindler [32]), and the Bloch-ball gyrovector structure (Chen–Ungar)—technical contributions are:

(1) **Metric-level conformal identity (qubit).** Extending the gyrovector identification of Chen–Ungar from algebra to Riemannian geometry, we show that on the qubit Bloch ball the Beltrami–Klein metric is conformal to the Bures metric with explicit factor $ds^2_{\mathrm{BK}} = 4\gamma^2(r)\, ds^2_{\mathrm{Bures}}$ (Eq. (7)).

(2) **Sharp boundary at $N \geq 3$.** We prove a conformal obstruction: for $N \geq 3$ the Bures metric is not conformally equivalent to any constant-curvature hyperbolic

model, using Weyl-tensor invariance together with an explicit sectional-curvature nonconstancy at the maximally mixed state (Theorem 1).

Secondary clarifications (useful but technically elementary) include the visibility-coordinate identity $I(V) = \gamma^2(V)$, and the Gudermannian relation $\arcsin(V) = \mathrm{gd}(\operatorname{arctanh} V)$ that relates Fisher–Rao arc-length to rapidity.

1.2 Conventions

Factor of 4. Throughout this paper, the quantum Fisher information (QFI) denotes the metric $g_{\mu\nu}^{\mathrm{QFI}}$ such that QFI $= 4\times$ Bures metric. The Fubini–Study metric on pure states equals one-quarter of the classical Fisher information metric [5]. All factor-of-4 relationships are stated explicitly where they arise.

Conformal relation. The central metric result $ds^2_{\mathrm{BK}} = 4\gamma^2 \cdot ds^2_{\mathrm{Bures}}$ uses the Bures metric. Equivalently, $ds^2_{\mathrm{BK}} = \gamma^2 \cdot ds^2_{\mathrm{QFI}}$.

Notation. V = visibility $= 2p - 1 \in (-1, 1)$ (open interval; regularity conditions exclude $V = \pm 1$ where Fisher information diverges). $r = |\vec{r}\,|$ = Bloch ball radial coordinate $= |V|$ for the 1D Bernoulli manifold. $\xi = \operatorname{arctanh}(V)$ = rapidity. $\gamma = 1/\sqrt{1 - V^2}$ = Lorentz factor. $I(V) = \gamma^2(V)$ throughout.

2 The Identity: Complete Derivation

2.1 Classical Fisher Information

For a Bernoulli distribution with parameter $p \in (0, 1)$, the Fisher information is $I(p) = 1/[p(1 - p)]$. Reparameterizing via $p = (1 + V)/2$ with Jacobian $dp/dV = 1/2$:

$$I(V) = I(p) \cdot \left(\frac{dp}{dV}\right)^2 = \frac{1}{p(1-p)} \cdot \frac{1}{4} = \frac{4}{1 - V^2} \cdot \frac{1}{4} = \frac{1}{1 - V^2} = \gamma^2(V). \tag{2}$$

2.2 Quantum Fisher Information

For a qubit state $\rho = (\mathbb{I} + \vec{r} \cdot \vec{\sigma})/2$ with Bloch vector $\vec{r}$, the quantum Fisher information with respect to the radial parameter $r = |\vec{r}\,|$ is:

$$F_Q(r) = \frac{1}{1 - r^2} = \gamma^2(r). \tag{3}$$

This equals $4\times$ the Bures metric radial coefficient: $g_{rr}^{\mathrm{Bures}} = 1/[4(1 - r^2)] = \gamma^2/4$.

3 The Geometry: Spherical Information, Hyperbolic Kinematics

3.1 The Classical 1D Bernoulli Manifold

The Fisher–Rao line element on the 1D Bernoulli manifold parameterized by V is:

$$ds^2_{\mathrm{FR}} = I(V)\, dV^2 = \frac{dV^2}{1 - V^2}. \tag{4}$$

The Fisher–Rao geodesic distance from $V = 0$ to V is $d_{\mathrm{FR}}(0, V) = \arcsin(V)$. The rapidity coordinate is $\xi = \operatorname{arctanh}(V)$; the two arc-length parameters are related by the Gudermannian identity $\arcsin(V) = \operatorname{gd}(\xi)$ with $V = \tanh \xi$.

3.2 The Bloch Ball with QFI Metric (Spherical—Hemisphere of S^3)

The Bures metric on the interior of the Bloch ball $B^3 = \{r < 1\}$ is:

$$ds^2_{\mathrm{Bures}} = \frac{dr^2}{4(1-r^2)} + \frac{r^2}{4}\, d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. This is the round metric on the northern hemisphere of S^3 of radius $\frac{1}{2}$ (constant positive sectional curvature $K = +4$), identified with the interior of the Bloch ball via the coordinate substitution $r = \sin \chi$ (equivalently $\chi = \arcsin r$). The Bures geodesic distance from the center to radius r is $d_{\mathrm{Bures}}(0, r) = \frac{1}{2} \arcsin(r)$; equivalently, the QFI geodesic distance (with $ds^2_{\mathrm{QFI}} = 4\, ds^2_{\mathrm{Bures}}$) is $d_{\mathrm{QFI}}(0, r) = \arcsin(r)$.

3.3 The Beltrami-Klein Metric on the Bloch Ball (Hyperbolic)

The Beltrami-Klein model of hyperbolic 3-space H^3 on the same open ball B^3 is:

$$ds^2_{\mathrm{BK}} = \frac{dr^2}{(1-r^2)^2} + \frac{r^2}{1-r^2}\, d\Omega^2. \tag{6}$$

This has constant negative sectional curvature $K = -1$. The geodesic distance is $d_{\mathrm{BK}}(0, r) = \operatorname{arctanh}(r)$.

3.4 The Bi-Metric Comparison

Dividing Eq. (6) by Eq. (5) component-wise:

Radial: $\dfrac{dr^2/(1-r^2)^2}{dr^2/[4(1-r^2)]} = \dfrac{4}{1-r^2} = 4\gamma^2(r).$

Angular: $\dfrac{r^2\, d\Omega^2/(1-r^2)}{r^2\, d\Omega^2/4} = \dfrac{4}{1-r^2} = 4\gamma^2(r).$

Both components yield the same conformal factor:

$$\boxed{ds^2_{\mathrm{BK}} = 4\gamma^2(r) \cdot ds^2_{\mathrm{Bures}}.} \tag{7}$$

This is a genuine conformal equivalence. Equivalently, $ds^2_{\mathrm{BK}} = \gamma^2 \cdot ds^2_{\mathrm{QFI}}$.

Remark (Ricci flow interpretation). The Beltrami–Klein metric g_{BK}, being Einstein with $\mathrm{Ric}(g_{\mathrm{BK}}) = -2\, g_{\mathrm{BK}}$, is a fixed point of volume-normalized Ricci flow. In three dimensions, the Weyl tensor vanishes identically and conformal flatness is controlled by the Cotton tensor; our conformal relation $ds^2_{\mathrm{BK}} = 4\gamma^2\, ds^2_{\mathrm{Bures}}$ is an explicit global identity, not an automatic consequence of dimension. This perspective also illuminates the $N \geq 3$ obstruction: in dimensions ≥ 4, the non-vanishing Weyl tensor provides a conformal invariant that obstructs equivalence to constant-curvature models.

4 Six Complementary Perspectives

4.1 Dependency Structure

The six perspectives are complementary viewpoints rather than strictly independent derivations. Perspectives 2–5 are closely related representations of the same underlying structure (the Bloch/gyrovector picture and the $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SO}^+(1,3)$ correspondence), while Perspectives 1 and 6 provide operational and information-geometric context. We order them by historical lineage and increasing abstraction.

4.2 Perspective 1: Stokes–Minkowski Polarization Optics

The Stokes parameters (S_0, S_1, S_2, S_3) form a Minkowski 4-vector. Mueller matrices preserving the Stokes cone are Lorentz transformations. For degree of polarization $V = |\vec{S}\,|/S_0$, the transformation from unpolarized to polarized is a Lorentz boost with $\beta = V$, yielding $I(V) = \gamma^2(V)$.

4.3 Perspective 2: Bloch Ball Gyrovector Algebra

Chen and Ungar [8, 81] proved that Bloch vectors compose via Einstein velocity addition. Chen, Fu, Ungar, and Zhao [127] provided the rapidity parametrization and hyperbolic-triangle interpretation of Bures fidelity. The Bloch ball under this composition is a gyrovector space—the natural algebraic framework for hyperbolic geometry. The identification $V \leftrightarrow \beta$ is an algebraic isomorphism.

4.4 Perspective 3: SLD Quantum Fisher Information

The symmetric logarithmic derivative (SLD) Fisher information for a qubit state gives $F_Q(r) = 1/(1-r^2) = \gamma^2(r)$. This is the Petz-distinguished monotone metric [33]: by the Petz classification, the SLD (Bures) metric is the minimal element in the family of quantum monotone metrics—it gives the tightest quantum Cramér–Rao bound and is operationally optimal for quantum state estimation.

4.5 Perspective 4: Burns–Greenfield–Dressel Measurement Dynamics

Burns, Greenfield, and Dressel [11] (*Quantum Stud.: Math. Found.* **13**, art. 10, 2026) established that the combined group of unitary evolution and non-unitary measurement backaction on a continuously monitored qubit is $\mathrm{SL}(2,\mathbb{C})$: unitary rotations $\to$ spatial rotations in $\mathrm{SO}(3)$; measurement-induced state changes $\to$ Lorentz boosts. We cite Burns–Greenfield–Dressel for the $\mathrm{SL}(2,\mathbb{C})$/Lorentz *group-action* correspondence; our metric identity is independent. Measurement strength maps to rapidity $\xi = \operatorname{arctanh}(V)$.

4.6 Perspective 5: Chentsov Uniqueness and the Petz Classification

Proposition 1 (Chentsov necessity of the γ^2 form)**.** *Let g be any Riemannian metric on the Bernoulli manifold $\mathcal{B} = \{\mathrm{Bernoulli}(p) : p \in (0,1)\}$ that is invariant under Markov* *morphisms and expressed in the visibility coordinate $V = 2p-1$. Then $g = c \cdot dV^2/(1-V^2)$ for some constant $c > 0$. With the standard normalization $c = 1$, $I(V) = \gamma^2(V)$.*

At the quantum level, the Petz classification [33] identifies a continuous family of monotone metrics. The SLD (Bures) metric is the *minimal* element—operationally optimal for quantum estimation via the quantum Cramér–Rao bound.

4.7 Perspective 6: Cramér–Rao Bound

The quantum Cramér–Rao bound gives $\mathrm{Var}(\hat{V}) \geq 1/[N \cdot F_Q(V)]$. Substituting $F_Q(V) = \gamma^2(V)$:

$$\mathrm{Var}(\hat{V}) \geq \frac{1 - V^2}{N} = \frac{1}{N\,\gamma^2(V)}. \tag{8}$$

As $V \to 1$, estimation precision diverges as γ^2—the same divergence as time dilation approaching the speed of light. The bound is saturable for qubits.

Remark (Quantum speed limit). The Mandelstam–Tamm quantum speed limit bounds the minimum time to evolve between distinguishable states: $\tau \geq \pi\hbar/(2\Delta E)$, where the distance is measured in the Bures metric [61, 62]. Since $ds_{\mathrm{BK}} = 2\gamma\, ds_{\mathrm{Bures}}$, the conformal factor directly rescales the speed limit between the two geometries: traversing a Bures arc-length ϵ requires minimum time $\propto \epsilon$, whereas the same displacement measured in Beltrami-Klein coordinates corresponds to $2\gamma\epsilon$, amplified by the Lorentz factor. The γ^2 divergence near purity ($V \to 1$) thus acquires an operational interpretation: the information-geometric "speed of light" imposes an asymptotic barrier on how quickly a nearly pure state can be distinguished from a pure state, paralleling the relativistic velocity bound.

5 The Group-Theoretic Bridge

5.1 The Isomorphism

The special linear group $\mathrm{SL}(2, \mathbb{C})$ is the double cover of the proper orthochronous Lorentz group: $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}^+(1, 3)$. This is a textbook result (Penrose–Rindler [32], Ch. 1). Qubit states transform under $\mathrm{SL}(2, \mathbb{C})$: unitary operations ($\mathrm{SU}(2) \subset \mathrm{SL}(2, \mathbb{C})$) are spatial rotations; non-unitary operations are boosts.

5.2 The Invariant Content of the Identity

The invariant content of $I(V) = \gamma^2$ is: the Chentsov-unique Fisher–Rao metric on the Bernoulli manifold, expressed in the coordinate that $\mathrm{SL}(2, \mathbb{C})$ identifies as a velocity-like parameter, takes the form of the squared Lorentz factor. Both sides are determined by the same group.

5.3 What the Identity Does NOT Claim

The identity does NOT claim: (a) that qubit state space "is" Minkowski spacetime; (b) that measurement "causes" relativistic effects; (c) that Fisher information replaces the energy-momentum tensor; (d) that the correspondence extends to gravitational physics.

5.4 The Curvature Clarification

The Bures metric on the 3D Bloch ball has constant sectional curvature $K = +4$ (hemisphere of S^3 of radius $\frac{1}{2}$); equivalently, the QFI metric ($= 4\times$ Bures) has $K = +1$. The Beltrami-Klein metric has $K = -1$ (H^3). The correspondence operates at the 1D level (no curvature ambiguity) and at the group level. The 3D metrics are related by conformal equivalence Eq. (7), not isometry.

The Gudermannian function provides the bridge: $\mathrm{gd}(\mathrm{arctanh}(V)) = \arcsin(V)$. This connects the hyperbolic geodesic distance $\mathrm{arctanh}(V)$ to the spherical geodesic distance $\arcsin(V)$.

6 The Measurement-Relativity Dictionary

Table 1: Measurement–Relativity Dictionary

Measurement Domain	**Relativity Domain**	**Bridge**
Visibility V	Velocity β	$V \leftrightarrow \beta$
Fisher info $I(V)$	γ^2	Eq. (1)
Rapidity ξ_{meas}	$\xi = \mathrm{arctanh}(\beta)$	Identical
Bures metric	(spherical)	Eq. (5)
Beltrami-Klein	Velocity space	Eq. (6)
Conformal factor $4\gamma^2$	—	Eq. (7)
Sequential measurements	Boost composition	Thomas–Wigner
$\mathrm{SL}(2,\mathbb{C})$ on Bloch ball	$\mathrm{Spin}^+(1,3)$ on spacetime	Exceptional isomorphism

7 The Entropic Identity and Holographic Connections

The von Neumann entropy of a qubit state with visibility V is $S = H((1+V)/2)$, where H is the binary entropy function. The *entropic identity* $S_{\mathrm{EE}} = H(V)$ has been proposed as an information-theoretic bridge; we present this as a structural analogy requiring further investigation.

7.1 Holographic Analogies (Structural, Not Proven)

Lashkari and Van Raamsdonk [16] proved (within the AdS/CFT setting) that canonical energy equals quantum Fisher information for holographic CFT states with AdS duals. If entanglement entropy plays the role of $H(V)$ and Fisher information plays the role of γ^2, the present framework suggests a structural parallel. We emphasize this is an analogy, not a derivation.

8 Cross-Domain Convergence

The identity $I(V) = \gamma^2(V)$ is not isolated. Four independent research programs have converged on the same $\mathrm{SL}(2,\mathbb{C})$/Lorentz/$\gamma^2$ structure.

8.1 Quantum Optics: Squeezing as Lorentz Boost (1988–2021)

Han, Kim, and Noz [93] established that squeeze operators generate $\mathrm{SU}(1,1) \cong \mathrm{SO}(2,1)$ transformations—the 2+1D Lorentz group. In $\mathrm{SU}(1,1)$ interferometry [94], QFI for phase estimation scales as γ^2. LIGO's squeezed vacuum injection [96] is a Lorentz boost on the vacuum state.

8.2 Condensed Matter: Quantum Geometric Tensor

Zanardi, Giorda, and Cozzini [97] proved that fidelity susceptibility diverges as the QFI metric coefficient at quantum phase transitions. For two-band systems, this takes the γ^2 form.

8.3 Stochastic Thermodynamics: Fisher Information as Entropy Production

Ito and Dechant [19] relate temporal Fisher information to Cramér–Rao-type speed limits for stochastic time evolution. In particular, temporal Fisher information controls bounds on rates of change of observables and relaxation in overdamped Langevin dynamics. We cite this work as an example of how Fisher-information quantities constrain nonequilibrium dynamics; we do not claim a universal identity equating entropy production to temporal Fisher information outside the specific conditions analyzed in a given model.

8.4 Holography: Fisher Information as Gravitational Energy

Lashkari and Van Raamsdonk [16] proved canonical energy = QFI for holographic CFTs with AdS duals.

Summary. These four programs each independently arrived at the same γ^2 or $\mathrm{SL}(2,\mathbb{C})$/Lorentz structure. The identity $I(V) = \gamma^2(V)$ is a compact algebraic signature that appears across these lines of work. The present paper provides the synthesis.

9 Relationship to Concurrent and Prior Work

Connection to relational quantum mechanics and quantum reference frames. The visibility parameter V is inherently relational: defined relative to a measurement basis and an observer. This aligns with the core thesis of relational quantum mechanics. Adlam and Rovelli [123] argue that "information is physical" in the context of cross-perspective links, and Di Biagio and Rovelli [100] formalize relative information using Shannon theory, defining relative facts through mutual information. The identity $I(V) = \gamma^2(V)$ provides a complementary geometric structure on this space of relative information states.

Giacomini, Castro-Ruiz, and Brukner [78] showed QM remains covariant under QRF changes, and Apadula et al. [58] extended this to Lorentz symmetry. Perche and Martín-Martínez [90] showed spacetime geometry can be recovered from quantum measurements. The conformal mapping $ds^2_{\mathrm{BK}} = 4\gamma^2 \cdot ds^2_{\mathrm{Bures}}$ can be interpreted as the information-geometric counterpart of switching between informational and kinematic descriptions.

10 Addressing Potential Objections

10.1 "It's just a Jacobian / coordinate artifact"

The identity $I(V) = \gamma^2(V)$ is the Chentsov-unique metric coefficient in the visibility coordinate. The choice of V is physically selected by the $\mathrm{SL}(2, \mathbb{C})$ group structure.

10.2 "The Fisher metric has the wrong signature"

The Fisher metric is positive-definite (Riemannian); the Minkowski metric is indefinite. The results operate at three levels: the group level ($\mathrm{SL}(2, \mathbb{C})$ generates both), the 1D metric level ($I(V) = \gamma^2$), and the conformal level. At none of these does signature play a role. Whether a Lorentzian extension exists is an open structural question for future work.

10.3 "Both spaces are spherical—where's the SR connection?"

Addressed in Section 5.4. The 3D geometries are NOT isometric. The correspondence operates at the 1D level and group level; the 3D metrics are related by conformal equivalence Eq. (7).

10.4 "This only works for qubits"

True as stated. However: (a) binary-outcome measurements are ubiquitous and operationally natural (threshold detectors, yes/no POVMs); (b) many physical settings reduce to effective qubits (two-band Hamiltonians, polarization optics, spin-$\frac{1}{2}$ subspaces); (c) the obstruction itself provides a precise boundary statement for when constant-curvature hyperbolic models fail.

Theorem 1 (Conformal Non-Equivalence for $N \geq 3$)**.** *For N-level quantum systems with $N \geq 3$, the Bures metric on the space of full-rank density matrices is* not *conformally equivalent to the Beltrami-Klein metric on any d-dimensional hyperbolic ball ($d = N^2 - 1$).*

Proof. The argument proceeds in five steps.

Step 1 (Conformal invariance of the Weyl tensor). The Beltrami-Klein metric on B^d has constant sectional curvature $K = -1$, hence vanishing Weyl tensor $W = 0$. For $d \geq 4$, the $(1, 3)$-Weyl tensor is a conformal invariant [109]: if $g_1 = \Omega^2 g_2$, then $W(g_1) = W(g_2)$. Since $d = N^2 - 1 \geq 8$ for $N \geq 3$, conformal equivalence would force $W(g_B) = 0$.

Step 2 (Einstein condition at the maximally mixed state). At $\rho_* = I/N$, the group $\mathrm{SU}(N)$ acts on $T_{\rho_*}\mathcal{D}_N^\circ \cong \mathfrak{su}(N)$ via the adjoint representation, which is irreducible (since $\mathfrak{su}(N)$ is simple; equivalently, the only Ad-invariant bilinear forms on $\mathfrak{su}(N)$ are multiples of the Killing form). Both g_B and Ric are $\mathrm{SU}(N)$-invariant symmetric $(0, 2)$-tensors, so by Schur's lemma: $\mathrm{Ric}(\rho_*) = \lambda\, g_B(\rho_*)$ for some $\lambda \in \mathbb{R}$.

Step 3 (Einstein + $W = 0$ forces constant curvature). At any Einstein point ($\mathrm{Ric} = \lambda g$) with $W = 0$, the standard decomposition of the Riemann tensor reduces to $R = \frac{\lambda}{d-1}\, g \owedge g$, the curvature tensor of a space of constant sectional curvature [109].

Step 4 (Nonconstant sectional curvature). Dittmann [53] proved that for $N > 2$ the Bures manifold of nonsingular density matrices is not of constant curvature and not locally symmetric; Dittmann [24] further showed curvature diverges near lower-rank boundaries. This abstract result is the load-bearing step; the following O'Neill computation is an explicit qutrit illustration in our conventions, not essential to the proof. We verify nonconstancy at ρ_* explicitly via the O'Neill submersion formula [110]: since the Bures metric arises from the Riemannian submersion $\pi : S^{2N^2-1} \to \mathcal{D}_N^\circ$, $\pi(W) = WW^\dagger$ [111], the sectional curvature at ρ_* for Bures-orthonormal $X, Y \in T_{\rho_*}\mathcal{D}_N^\circ$ is

$$K_B(X,Y) = 1 + \tfrac{3N^3}{64}\Big(-\mathrm{Tr}([X,Y]^2)\Big). \tag{9}$$

For $N = 3$: commuting pairs (Cartan subalgebra elements, $[H_1, H_2] = 0$) give $K_B = 1$, while non-commuting root vectors with $[\lambda_1, \lambda_2] = 2i\lambda_3$ give $K_B > 1$. Hence sectional curvature at ρ_* is not constant: distinct 2-planes yield distinct values of K_B, confirming Dittmann's global result pointwise.

Step 5 (Contradiction via Riemann decomposition). At the Einstein point ρ_* (Step 2), the Schouten tensor takes the form $S = \frac{\lambda}{2(d-1)} g_B$, so its Kulkarni-Nomizu contribution to the Riemann tensor is $S \owedge g = \frac{\lambda}{d-1} g_B \owedge g_B$—the curvature tensor of a constant-curvature space. The standard decomposition $R = W + S \owedge g$ (valid at any point; see Besse [109], §1.G) then yields $W = R - \frac{\lambda}{d-1} g_B \owedge g_B$. Now suppose $W(\rho_*) = 0$. Then the full Riemann tensor at ρ_* would reduce to $R = \frac{\lambda}{d-1} g_B \owedge g_B$, which is the curvature tensor of constant sectional curvature $K = \lambda/(d-1)$; in particular, *every* 2-plane at ρ_* would have the same sectional curvature. But Step 4 establishes that sectional curvatures at ρ_* are *not* all equal (Dittmann [53, 24]). This is a contradiction, so $W(g_B)(\rho_*) \neq 0$, contradicting the assumption of conformal equivalence from Step 1.

For $N = 2$: every pair of Pauli matrices yields $K_B = 4$ (constant curvature = 4), consistent with the known Uhlmann hemisphere $\frac{1}{2}S^3$ [111]. The Weyl tensor vanishes identically in dimension 3, so no obstruction arises. □

Remark (Why qubits are special). The obstruction traces to the representation theory of curvature tensors. For SU(2), the space of invariant algebraic curvature tensors on $\mathfrak{su}(2)$ is 1-dimensional—only the constant-curvature form $g \owedge g$ survives—ensuring conformal flatness. For SU(3), this space is 2-dimensional: the Lie bracket curvature tensor $R_2(X, Y, Z, W) = \langle [X, Y], [Z, W] \rangle$ provides a second, independent component with nonvanishing Weyl part. The O'Neill formula guarantees this component has positive coefficient in the Bures curvature. This dimension jump—from 1 to 2 in the invariant curvature space—is the structural root of the qubit/qutrit dichotomy. This result strengthens the Lie algebra obstruction $\mathfrak{su}(3) \not\cong \mathfrak{spin}(1, d)$ previously noted by the author, which is necessary but not sufficient for conformal non-equivalence.

This theorem sharpens the scope of the Bloch-gyrovector program by showing that the metric-level constant-curvature hyperbolic model does not extend to $N \geq 3$ density-matrix manifolds; the obstruction is geometric (a nonvanishing Weyl tensor), not merely algebraic.

Remark (Coleman–Mandula parallel). The conformal non-equivalence resonates with the Coleman–Mandula theorem [92], which requires spacetime and internal symmetries to factorize. Within the present framework, spacetime structure is qubit measurement structure ($\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}^+(1, 3)$), while internal gauge symmetries such as color SU(3) correspond to Hilbert space dimensions whose measurement geometry cannot be conformally mapped to any constant-curvature spacetime model. Whether this reflects a deeper connection remains open.

11 Experimental Tests

11.1 Cramér–Rao Saturation on Single Qubits

Prepare a qubit in state $\rho(V) = (\mathbb{I} + V\sigma_z)/2$. Perform N projective measurements. The estimator variance should saturate: $\mathrm{Var}(\hat{V}) = (1 - V^2)/N = 1/(N\gamma^2)$. Achievable with maximum-likelihood estimation.

11.2 Two-Band Quantum Geometric Tensor

Measure the full QGT in a two-band system, following Kang et al. [28] (*Nature Physics*, 2025) and Kim et al. [29] (*Science*, 2025). For two-band Hamiltonians, the QGT reduces to the qubit QFI metric. The γ^2 scaling with band gap closing ($V \to 1$) is a direct test.

11.3 Thomas–Wigner Rotation for Non-Collinear Measurements

Perform two sequential non-collinear projective measurements on a qubit. After each measurement, the Bloch vector undergoes a Lorentz-like boost. Composing two non-collinear boosts produces a Thomas–Wigner rotation. The predicted rotation angle Ω is determined by the measurement visibilities V_1, V_2 and the angle θ between measurement axes. A convenient closed form (for rapidities $\xi_i = \mathrm{arctanh}(V_i)$ and axis angle θ) is the standard Thomas–Wigner formula for composing non-collinear boosts (see, e.g., Rhodes and Semon [46] or Yeh [67]):

$$\tan\frac{\Omega}{2} = \frac{\tanh(\xi_1/2)\tanh(\xi_2/2)\sin\theta}{1 + \tanh(\xi_1/2)\tanh(\xi_2/2)\cos\theta}. \tag{10}$$

Quantitative example. For two orthogonal measurements ($\theta = 90°$) with equal visibility $V_1 = V_2 = 0.6$ (so $\xi_1 = \xi_2 = \mathrm{arctanh}(0.6) \approx 0.693$): $\tanh(\xi/2) = \tanh(0.347) \approx 0.333$, giving $\tan(\Omega/2) = 0.333^2 \approx 0.111$, hence $\Omega \approx 12.7°$. For $V = 0.9$: $\xi \approx 1.472$, $\tanh(\xi/2) \approx 0.627$, giving $\tan(\Omega/2) \approx 0.393$, hence $\Omega \approx 42.9°$. The rotation grows non-linearly with visibility, vanishes at $V = 0$ (maximally mixed), and as $V \to 1$ (pure-state limit) the Bloch-sphere rotation angle approaches $\Omega \to \theta$; the corresponding spinor/SU(2) phase is $\Omega/2 \to \theta/2$, consistent with the usual double-cover/Berry-phase half-angle. The deviation from the Pancharatnam prediction at intermediate V is the measurable Beltrami-Klein signature.

Concrete protocol. Prepare a qubit in a mixed state ρ_0 with Bloch vector of magnitude r (purity parameter). Perform partial measurements along $\hat{z}$, then $\hat{x}$ (or any two non-collinear axes). Reconstruct the final state via quantum state tomography. This 'boost composition' picture applies to symmetric/Lüders measurements where the relevant Kraus operators can be taken positive; general Kraus maps include additional unitary rotations. The key testable prediction: the acquired geometric rotation depends on the initial purity r according to the hyperbolic (Wigner) formula, not the spherical (Pancharatnam) formula that applies only at $r = 1$ (pure states). At $r \to 0$, the rotation vanishes; at intermediate r, the deviation from Pancharatnam's formula is the measurable signature of the Beltrami-Klein geometry. This is achievable on superconducting qubit platforms with dispersive readout (measurement efficiency $\eta > 0.5$, tomographic precision $\sim 1\%$ in Bloch vector components, $\sim 10^4$ shots per setting).

11.4 Discriminating Test: γ^2 Scaling of Estimation Precision

Sweep visibility V from 0 to 0.99 and measure estimation precision. If $I(V) = a/(1-V^2)^b$, the generic statistical explanation predicts $b = 1$, $a = 1$ (simple Bernoulli Fisher information). The present framework predicts $b = 1$ exactly, but with the specific interpretation that the scaling reflects $\mathrm{SL}(2,\mathbb{C})$ structure. A deviation $b \neq 1$ would falsify the identity at its mathematical level.

12 Status of Claims

Table 2: Claim status in this paper

Claim	**Basis**	**Status here**
$I(V) = \gamma^2(V)$ (classical)	Direct computation in Sec. 2.1	Derived here
$F_Q(r) = \gamma^2(r)$ (qubit radial QFI)	Standard qubit QFI/Bures formulas (Sec. 2.2)	Recalled (cited)
$ds^2_{\mathrm{BK}} = 4\gamma^2(r)\, ds^2_{\mathrm{Bures}}$	Component-wise metric comparison (Sec. 3.4)	Derived here
Chentsov forces Fisher–Rao form in V	Chentsov theorem + reparametrization (Prop. 1)	Derived here
Bures/SLD is a distinguished monotone metric	Petz classification context (Sec. 4.4–4.6)	Prior work (cited)
Measurement backaction generates boosts	Continuously monitored qubit dynamics (Burns–Greenfield–Dressel)	Prior work (cited)
Thomas–Wigner rotation for non-collinear updates	Cartan/boost composition + measurement model assumptions	Testable prediction
Conformal non-equivalence for $N \geq 3$	Weyl invariance + Einstein at maximally mixed + Dittmann nonconstant curvature (O'Neill check optional; Thm. 1)	Derived here
Gudermannian bridge $\arcsin(V) = \mathrm{gd}(\operatorname{arctanh} V)$	Elementary identity used in Sec. 5.4	Derived here
Coleman–Mandula parallel	Heuristic analogy only	Speculative

13 Discussion

The identity $I(V) = \gamma^2(V)$ is algebraically elementary. Its significance comes from three sources: Chentsov uniqueness at the classical level and the Petz classification at the quantum level (jointly establishing that γ^2 is the unique operationally optimal metric coefficient, up to normalization), the $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}^+(1,3)$ group structure that provides the physical bridge to Lorentz kinematics, and consistency tests on existing qubit platforms.

Completing Chen–Ungar. Chen and Ungar [8, 80, 81, 82] identified the Bloch ball as a Beltrami-Klein model of hyperbolic geometry. This paper extends their algebraic identification to the metric level: the conformal equivalence $ds^2_{\rm BK} = 4\gamma^2 \cdot ds^2_{\rm Bures}$ quantifies the precise relationship between the two natural geometries of the Bloch ball, with the conformal factor being the Fisher information = Lorentz factor identity. Kim, Noz, and collaborators [93, 95] proved the group theory for squeezing (1988). Lévay [83] identified a connection between Thomas–Wigner rotation and Uhlmann parallel transport for qubits (2004), working with abstract geodesic triangles in the Bures geometry; the present paper reframes this in terms of operationally accessible sequential measurements parametrized by visibility, yielding concrete numerical predictions (Eq. 10). Burns, Greenfield, and Dressel [11] proved the measurement dynamics (2026). Each established a different facet of the same structure. The present paper provides the scalar identity and conformal equivalence linking all facets.

Convergence with quantum reference frames. The QRF program [58, 59] has independently demonstrated that switching between quantum measurement perspectives is implemented by Lorentz boosts. Apadula, Castro-Ruiz, and Brukner [105] define quantum Lorentz transformations for spin-$\frac{1}{2}$ systems in a perspective-neutral QRF framework, and the generating-function formalism of Chen [106] provides explicit Bloch-sphere formulas linking fidelity, quantum metric, and Fisher information. The correspondence $I(V) = \gamma^2$ provides the information-geometric invariant underlying these QRF transformations—a connection that, to our knowledge, has not been made explicit.

Modular flow and the Bisognano–Wichmann theorem. The identity $I(r) = \gamma^2(r)$ acquires a structural interpretation through Tomita–Takesaki modular theory. For a qubit thermal state $\rho(\beta) = e^{-\beta H}/\mathrm{tr}\, e^{-\beta H}$ with $H = \frac{\varepsilon}{2}\sigma_z$, the Bloch vector has magnitude $r = \tanh(\beta\varepsilon/2)$—the functional form of the velocity-rapidity relation $v = \tanh\eta$, with $\eta = \beta\varepsilon/2$ as rapidity. The squared Lorentz factor becomes $\gamma^2 = \cosh^2(\beta\varepsilon/2)$, which our theorem identifies as the Fisher information. The Bisognano–Wichmann theorem [117] states that the modular flow of the Minkowski vacuum restricted to a Rindler wedge is the one-parameter group of Lorentz boosts. In this light, the Bures metric corresponds to the proper-distance metric, the Beltrami-Klein metric to the coordinate-distance metric, and the conformal factor $4\gamma^2$ mirrors the structure of the Tolman redshift between them—though this identification is structural and analogical, since the Bisognano–Wichmann theorem strictly applies to quantum field theories on Rindler wedges, not to finite-dimensional qubits. The $N \geq 3$ failure maps to the well-known fact that modular Hamiltonians for higher-dimensional systems are generically non-local [118]—the clean boost structure is lost.

Thermodynamic length and the third law. The conformal identity acquires thermodynamic content through the thermodynamic length framework [119, 120]. Since $ds_{\rm BK} = 2\gamma \cdot ds_{\rm Bures}$, the Beltrami-Klein line element is the Bures line element weighted by twice the Lorentz factor. For a radial process from the maximally mixed state to purity r: the Bures geodesic distance is $d_{\rm Bures}(0, r) = \frac{1}{2}\arcsin(r)$ (consistent with the hemisphere of radius $\frac{1}{2}$), the QFI geodesic distance is $d_{\rm QFI}(0, r) = \arcsin(r)$, and the hyperbolic distance is $d_{\rm BK}(0, r) = \mathrm{artanh}(r)$. These are linked by the Gudermannian: $\mathrm{gd}(\mathrm{artanh}(r)) = \arcsin(r)$. The divergence $d_{\rm BK} \to \infty$ as $r \to 1$ is the geometric manifestation of the third law of thermodynamics: reaching a pure state requires infinite hyperbolic distance, just as reaching the speed of light requires infinite rapidity. The conformal factor $\gamma^2 = (1 - r^2)^{-1}$—the Fisher information—is a thermodynamic Lorentz factor quantifying the cost amplification near purity.

Connections to emergent gravity and holographic complexity. Berglund et al. [101] argue that quantum gravity may violate Chentsov's theorem assumptions, proposing a "generally covariant information geometry" where the Fisher metric becomes dynamical. The conformal equivalence $ds^2_{\mathrm{BK}} = 4\gamma^2 \cdot ds^2_{\mathrm{Bures}}$ is a concrete, exact instance of Fisher-metric structure naturally producing relativistic geometry. Bianconi [102] derives modified Einstein equations from quantum relative entropy between spacetime metrics, producing an emergent cosmological constant. Gerbershagen et al. [103] identify a holographic dual of the Bures metric in AdS/CFT, computing its bulk geometric interpretation within Susskind's complexity=volume paradigm. Remarkably, Brown and Susskind [115] study the complexity geometry of a single qubit and find regions of negative curvature resembling hyperbolic geometry on the Bloch ball—precisely the structure our conformal equivalence explains: the Beltrami-Klein metric provides the constant-curvature hyperbolic model that the Bures metric approximates conformally. Combined, these results suggest that the qubit serves as the simplest boundary system whose information geometry encodes a bulk hyperbolic geometry, and that the conformal factor γ^2 may play the role of a local scale relating boundary Fisher information to bulk distance.

Experimental outlook. Recent experiments have brought the quantum geometric tensor and Fisher information into the laboratory. Zhao et al. [107] verify QFI-coherence factorization experimentally for qubit and qutrit systems, providing a direct experimental context for the γ^2 scaling. Sala et al. (*Science* **389**, 822, 2025) report the first direct measurement of the quantum metric in materials via spin-momentum-locked electrons, using Bloch sphere pseudospin mapping. The γ^2 divergence as band gap closes ($V \to 1$) constitutes a testable prediction. Melo et al. [108] introduce conditional quantum Fisher information at the trajectory level, bridging quantum information geometry and stochastic thermodynamics on exactly the Bloch ball geometry where the conformal equivalence lives. Oancea et al. [104] generalize the quantum geometric tensor using sub-bundle geometry, with an explicit example of Dirac fermions on a hyperbolic plane demonstrating that spatial curvature modifies quantum geometry—directly contextualizing the conformal equivalence within modern QGT formalism.

14 Conclusion

We have shown that Fisher information of a binary quantum measurement equals the squared Lorentz factor: $I(V) = \gamma^2(V)$. We have proved that this identity reflects a conformal equivalence between the two natural geometries of the Bloch ball: $ds^2_{\mathrm{BK}} = 4\gamma^2 \cdot ds^2_{\mathrm{Bures}}$. We have established that this structure is algebraically special to qubits: the Weyl tensor of the Bures metric is nonvanishing for $N \geq 3$, providing a curvature obstruction to conformal equivalence with any constant-curvature model. And we have identified the Thomas–Wigner rotation as a testable prediction of the framework for sequential non-collinear measurements.

These results complete and extend the identification of the Bloch ball with the Beltrami-Klein model of hyperbolic geometry initiated by Chen and Ungar [8]. The conformal factor $\gamma^2 = I(V)$ is the specific scalar relating their algebraic observation to the information-geometric structure of quantum measurement. That four independent research programs in physics—quantum optics, condensed matter, stochastic thermodynamics, and holography—arrived at this same structure independently suggests the observation has depth beyond its algebraic simplicity. Recent work by Brody and Hughston [125] connecting decoherence to information gain on the Bloch ball, and by Caticha and Saleem [126] deriving quantum mechanics from entropic dynamics on Fisher-Rao manifolds within a relational framework, provides further evidence that the Fisher-metric geometry of qubits carries irreducible physical content.

Several open problems invite further investigation. Whether the conformal equivalence admits a Lorentzian extension recovering indefinite signature from information geometry remains an open structural question. The identity $I(V) = \gamma^2$ suggests an energy-precision trade-off $E_{\text{meas}} \cdot \text{Var}(V) \geq kT \ln 2$, unifying Landauer's erasure limit with the Cramér–Rao bound; rigorous derivation of this conjectured relation is left for future work. Whether the qutrit obstruction's resonance with Coleman–Mandula [92] reflects a deeper connection between measurement geometry and the Standard Model's symmetry structure is unknown.

Rovelli's relational quantum mechanics [122] postulates that there is a maximum amount of relevant information extractable from a system, and that quantum states encode relational information between systems [123, 124]. The identity $I(r) = \gamma^2(r)$ provides a geometric realization of this constraint: the information-geometric structure of a single qubit inherits the conformal geometry of relativistic velocity space, with the Lorentz factor encoding the relational distinguishability between quantum states.

Wheeler's "it from bit" postulated that spacetime geometry derives from binary quantum measurements, but left this as philosophical aspiration [75]. The identity $I(V) = \gamma^2(V)$ provides a restricted instance of this vision: on the qubit state space, the squared Lorentz factor—the signature of relativistic geometry—equals the Fisher information of binary measurements. That the conformal equivalence fails for $N \geq 3$ systems is consistent with Wheeler's emphasis on binary alternatives as geometrically distinguished, though whether this reflects a deeper principle remains open.

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Data Availability
No datasets were generated. All results are analytical.

Use of AI Tools
AI-assisted tools were used during literature exploration, drafting, and iterative refinement. The author takes full responsibility for all claims, derivations, and citations.

Author Contributions
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